Local Dimension of self-similar measures and similarity dimension

Let $\Phi = (\phi_i)_{i=1}^{m}$ be an IFS on $\mathbb{R}$.

Let $p = (p_1, \dots, p_m)$ be a probability vector.

If Φ satisfies the Strong Separation Property then

$$\dim_{\mathcal{H}}(\Lambda) = \text{sim-dim}(\Phi)$$

Let $\mu_p = (p_1, \dots, p_m)^{\mathbb{N}}$ on Σ_1

$\Big($ For $j \in \Sigma_n$ $\mu_p([j]) = \prod_{k=1}^{n} p_{j_k}$ $\Big)$

Let $\nu = \Pi(\mu_p)$ where $\Pi : \Sigma_1 \to \Lambda$ is the natural projection.

Claim: Since Φ satisfies SSP,

$$D_\nu(x) = \frac{-\sum_{i=1}^{m} p_i \log(p_i)}{-\sum_{i=1}^{m} p_i \log(r_i)} \quad \text{for } \nu\text{-a.e. } x.$$

(i.e. ν is exact dimensional).

Sketch of proof of claim:

Since Φ satisfies SSP, for every $x \in \mathrm{supp}(\nu)$ $\exists!\ \underline{i} \in \Sigma_1$ s.t. $\pi(\underline{i}) = x$.

For μ_p-a.e. $\underline{i}$

If $n_k(n) := \#\{\ell \in \{1, \dots, n\} \mid i_\ell = k\}$.

Law of Large numbers implies that for n large enough,

$$n_k(n) \approx p_k n.$$

also, SSP implies that

Note that $\pi_n(\underline{i}) = (i_1, \dots, i_n)$

and $r_{\pi_n(\underline{i})} := \prod_{\ell=1}^{n} r_{i_\ell} = \prod_{j=1}^{m} r_j^{n_j(n)}$

$$\nu(B(x, r_{\pi_n(\underline{i})})) \sim \prod_{\ell=1}^{n} p_{i_\ell} = \prod_{j=1}^{m} p_j^{n_j(n)}$$

Finally

$$D_\nu(x) = \lim_{n\to\infty} \frac{\log(\nu(B(x, r_{\pi_n(\underline{i})})))}{\log(r_{\pi_n(\underline{i})})}.$$

$$= \lim_{n\to\infty} \frac{\sum_{j=1}^{m} n_j(n) \log(p_j)}{\sum_{j=1}^{m} n_j(n) \log(r_j)}$$

$$= \lim_{n\to\infty} \frac{n\sum p_j \log(p_j)}{n\sum p_j \log(r_j)} .$$

$$= \frac{\sum p_j \log(p_j)}{\sum p_j \log(r_j)}$$

This computation leads to a notion of similarity dimension for self-similar measures.

<u>Def:</u>
Let ν be a self-similar measure corresponding to the self-similar IFS, Φ, with contraction ratios, $\{r_i\}_{i=1}^m$, and probability vector $p = (p_1, \dots, p_m)$. The <u>similarity dimension of ν</u> is defined by

$$\text{sim-dim}(\nu) := \frac{-\sum p_i \log(p_i)}{-\sum p_i \log(r_i)} = \frac{h_p}{\chi^p_r}$$

Here $h_p := -\sum_{i=1}^{m} p_i \log(p_i)$

is the entropy of the probability vector p and

$$\chi^p_r := -\sum_{i=1}^{m} p_i \log(r_i)$$

is the Lyapunov exponent (average contraction ratio).

Example

If $p = (r_1^s, \ldots, r_m^s)$

then ν is the natural measure on Λ and $s = \text{sim-dim}(\Phi)$

$$\text{sim-dim}(\nu) = \frac{h_p}{\chi^p_r} = \frac{-\sum p_i \log(p_i)}{-\sum p_i \log(r_i)}$$

$$= \frac{-\sum p_i \log(r_i^s)}{-\sum p_i \log(r_i)} = s.$$

Def: (Ergodic)

Let μ be a Borel probability measure on $\Sigma_1 := \{1, \dots, m\}^{\mathbb{N}}$ and let σ be the left shift

a.) We say that μ is a <u>σ-invariant measure</u> if $\mu(\sigma^{-1}A) = \mu(A)$ for all Borel sets $A \subset \Sigma_1$

b.) A Borel set $E \subset \Sigma_1$ is <u>σ-invariant</u> if $\sigma^{-1}(E) = E$.

c.) The measure μ is <u>ergodic</u> if for every invariant set E, $\mu(E) = 0$ or 1

Claim:

If $\mu = \{p_1, \dots, p_m\}^{\mathbb{N}}$, on Σ_1 then μ is σ-invariant and ergodic.

Def: Fix $m \geq 2$, $d \geq 1$. If Φ is an IFS If μ is a σ-invariant ergodic Borel probability measure on Σ_1, then the push-forward measure $\nu := \pi\mu$ is called an <u>ergodic invariant measure</u> for Φ

Thm: (Feng and Hu)

Every ergodic invariant measure for a self-similar IFS is exact dimensional

(Note that this does not require SSP or OSC).

A more precise statement:

Projection entropy

Consider the following (uncountable) partition of Σ_1:

$$Q = \{q\} \quad q \cap q' = \phi,$$

For $\underline{i} \in \Sigma_1$
$Q(\underline{i}) = q$ s.t. $\underline{i} \in q$.

Define

$$Q(\underline{i}) := \pi^{-1}(\pi(\underline{i})) \quad \text{where } \pi : \Sigma_1 \to \Lambda$$

is the natural projection.

For a.e. $\underline{i} \in \Sigma_1$, $\exists$ probability measure

$$\mu_{\underline{i}}^{Q} \quad \text{defined on } Q(\underline{i})$$

$\left(\text{If } Q \text{ is a finite partition, } \mu_{\underline{i}}^{Q}(E) = \dfrac{\mu(E \cap Q(\underline{i}))}{\mu(Q(\underline{i}))} \right)$

such that

- For $A \subset \Sigma_1$ Borel, $\underline{i} \mapsto \mu_{\underline{i}}^{Q}(A)$ is $\hat{Q}$-measurable

(where $\hat{Q}$ is the smallest σ-algebra containing Q)

- $$\mu(U) = \int_Q \mu_{\underline{i}}^{Q}(U) \, d\mu_Q(Q(\underline{i}))$$

where, for $H \subset Q$, $\mu_Q(H) = \mu(Q^{-1}(H))$

Thus $\mu(U) = \int_{\Sigma_1} \mu_{\underline{i}}^{Q}(U) \, d\mu(\underline{i})$.

<u>Example:</u> Consider $p = (p_1, p_2, p_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

$\underline{\Phi} = (\phi_1, \phi_2, \phi_3)$ $\phi_i : \mathbb{R} \to \mathbb{R}$

$$\phi_1(x) = 1/3 \, x$$
$$\phi_2(x) = \frac{1}{2} x$$
$$\phi_3(x) = \frac{1}{3} x + \frac{2}{3}.$$

Let $\underline{i} = (1, 1, 1, \ldots, 1, \ldots)$.

Then $\Pi(\underline{i}) = 0 \in \mathbb{R}$,

$$\Pi^{-1}(\Pi(\underline{i})) = \left\{ \underline{j} \in \Sigma_1 \;\middle|\; \begin{array}{l} j_m = 1 \text{ or } 2 \\ \text{for } m \text{ large enough} \end{array} \right\}.$$
$$= Q(\underline{i})$$

Then

$$\mu^{Q}_{\underline{i}} = \left\{\tfrac{1}{2}, \tfrac{1}{2}, 0\right\}^{\mathbb{N}}.$$

If $\underline{k} = (3, 3, 3, \ldots, 3, \ldots)$

Then $\pi(\underline{k}) = 1$

and

$$\pi^{-1}(\pi(\underline{k})) = \underline{k}$$

$$\Rightarrow \quad \mu^{Q}_{\underline{k}} = \delta_{\underline{k}}$$

If $\underline{\ell} = (2, 2, 2, 3, 3, 3, 3, \ldots, 3, \ldots)$

then

$$\pi^{-1}(\pi(\underline{\ell})) = \left\{ \underline{j} \;\middle|\; \begin{array}{l} j_1, j_2, j_3 = 1 \text{ or } 2 \\ j_m = 3 \text{ for } m \geq 4 \end{array} \right\}.$$

$$\mu^{Q}_{\underline{\ell}} = \left\{\tfrac{1}{2}, \tfrac{1}{2}, 0\right\}^{3}$$

$$= \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \sum_{j_3=1}^{2} \frac{1}{8} \delta_{\underline{j}}$$

Note that

$$\mu_{\underline{i}}^{Q}([i_1]) = \mu_{\underline{i}}^{Q}([1]) = \frac{1}{2}.$$

for all $\underline{j} \in Q(\underline{i})$

$$\mu_{\underline{j}}^{Q}([j_1]) = \frac{1}{2}.$$

$$\mu_{\underline{k}}^{Q}([k_1]) = \mu_{\underline{k}}^{Q}([3]) = \delta_{\underline{k}}([3]) = 1.$$

$$\mu_{\underline{l}}^{Q}([l_1]) = \mu_{\underline{l}}^{Q}([2]) = \frac{1}{2}.$$

$\Rightarrow$ $\mu_{l}^{Q}([l_1])$ measures the amount of overlap at stage 1.

We will see that this "first step" computation is enough to measure how much the Hausdorff dimension decreases from the similarity dimension due to overlap.

(Feng and Hu) Restated.

Thm: Let $p = (p_1, \dots, p_m)$, $\Phi = (\phi_1, \dots, \phi_m)$ be an IFS on $\mathbb{R}$. If μ_p on Σ is $\mu_p = \{p_1, \dots, p_m\}^{\mathbb{N}}$ and $\nu_p = \Pi_* \mu_p$. Then ν_p is exact dimensional and

$$\dim_H(\nu_p) = \frac{h_p - \left(-\int \log\left(\mu_{\underline{i}}^{Q}([i_1])\right) d\mu_p(\underline{i})\right)}{\chi_r^p}.$$