Steps of Proof of Hochman's Result

① Reduce to Entropy

For $n \in \mathbb{Z}_+$, let $D_n = \{ [\frac{k}{2^n}, \frac{k+1}{2^n}) \mid k \in \mathbb{Z} \}$

For μ a finite measure on $\mathbb{R}$ and $\mathcal{E}$ a countable partition of $\mathbb{R}$, then

$$H(\mu, \mathcal{E}) := -\sum_{E \in \mathcal{E}} \mu(E) \log(\mu(E))$$

and $H_n(\mu) := \frac{1}{n} H(\mu, D_n)$.

If for some probability vector $(p_1, \ldots, p_m)$

$$\mu(A) = \sum_{i=1}^{m} p_i \, \mu(\phi_i^{-1}(A))$$

then μ is a self-similar measure

Thm (Young '82, Feng-Hu '09)

If μ is a finite self-similar measure, the $\dim_H(\operatorname{supp} \mu) = \lim_{n \to \infty} \frac{H_n(\mu)}{\log 2}$

Thus if

μ is the measure supported on Λ satisfying

$$\mu = \sum r_i^s \, \mu \circ \phi_i^{-1}$$

the

$$\lim_{n \to \infty} \frac{H_n(\mu)}{-\log(2)} < s.$$

(2) Uniformize

$\dim_{\mathcal{H}}(\Lambda) <$ sim-dim(Φ)

implies

$\exists\, \Phi'$ s.t. if Λ' is the attractor of Φ' then

a.) $\Lambda' \subset \Lambda$

b.) for each $\phi_j' \in \Phi'$ $r_j = r$

c.) $\dim_{\mathcal{H}}(\Lambda') <$ sim-dim(Φ')

d.) For each j $\exists\, \underline{i}_j \in \Sigma_1^{*}$ s.t.

$$\phi_j' = \phi_{\underline{i}_j}$$

③ Disretize

Assuming $r_j = r_i \quad \forall i,j \in \{1, \dots, m\}$

$$\mu \approx \sum_{\underline{i} \in \Sigma_n'} r^{n s} \delta_{\phi_{\underline{i}}(0)} = \nu_n .$$

For m s.t. $2^{-m} \sim r^n$,

IF for each $q \in \mathcal{D}_m$, $\exists$ at most one $\underline{i} \in \Sigma_n'$ s.t. $\phi_{\underline{i}}(0) \in \mathcal{D}_m$,

then

$$\dim_{\mathcal{H}}(\mu) \approx \frac{-1}{\log 2} H_m(\nu_n) = \text{sim-dim}(\Phi).$$

But $\dim_{\mathcal{H}}(\mu) \neq \text{sim-dim}(\Phi)$

$\Rightarrow \exists \underline{i}, \underline{j} \in \Sigma_n'$ s.t. $|\phi_{\underline{i}}(0) - \phi_{\underline{j}}(0)| < r^n$.

④ Last Step

For any $k \in \mathbb{Z}_+$

$$H_{kn}(\nu_n) - H_n(\nu_n) \xrightarrow{n \to \infty} 0$$

$\min_{\underline{i}, \underline{j}} |\phi_{\underline{i}}(0) - \phi_{\underline{j}}(0)| < r^{kn}$ for every k.

for n large enough.

We first need to establish more background on the dimension of measures.

Local Dimension

Let μ be a Borel measure and $x \in supp(\mu)$. The local dimension of μ at x is defined by

$$D_\mu(x) := \lim_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log(r)}.$$

(If the limit exists)

Def: A finite Borel measure is <u>exact dimensional</u> of dimension α if the local dimension $D_\mu(x)$ exists and is equal to α for μ-a.e. x.

Lemma: For every finite Borel measure, μ, $x \in supp(\mu)$ and $(r_n)_{n=1}^\infty$ s.t. $r_n \to 0$, r_n decreasing and $\frac{\log(r_{n+1})}{\log(r_n)} \xrightarrow{n\to\infty} 1$

$$\liminf_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log(r)} = \liminf_{n\to\infty} \frac{\mu(B(x,r_n))}{\log(r_n)}$$

$$\limsup_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log(r)} = \limsup_{n\to\infty} \frac{\mu(B(x,r_n))}{\log(r_n)}$$

Example: $\mu = \mathcal{L}^2|_{[0,1]^2}$

Then $\mu(B(x,r)) \sim r^2$ for $x \in [0,1]^2$ and r small enough.

$\Rightarrow D_\mu(x) = 2$ for all $x \in [0,1]^2$.

Example:

$$\mu = \mathcal{L}^1|_{[0,1]} + \delta_{\{2\}}.$$

Example:

$$\mu = \sum_{\frac{p}{q} \in \mathbb{Q} \cap [0,1]} \frac{1}{q^{100}} \delta_{\frac{p}{q}}$$

For all $x \in [0,1]$,

$$\mu(B(x,r)) \gtrsim r^{50}$$

For $\frac{p}{q} \in \mathbb{Q}$, $\mu(B(\frac{p}{q}, r)) \sim q^{-100}$ for r small enough

$\Rightarrow D_\mu(\frac{p}{q}) = 0$

and for a.e. $x \in [0,1]$

$$\limsup_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log(r)} \leq 50$$

Since

$$\mu(B(x,\tfrac{1}{n})) \leq \sum_{k > n^{1/2}} \sum_{\ell=1}^{k/n} \frac{1}{k^{100}} \lesssim n^{-25}$$

$$\Rightarrow \liminf_{r\to 0} \frac{\mu(B(x,r))}{\log(r)} \geq 25.$$

Therefore, only calculating local dimension can be misleading.

Def: Let μ be a Borel measure on a metric space

$$\dim_{\mathcal{H}}(\mu) := \inf\{\dim_{\mathcal{H}}(E) \mid \mu(E^c)=0\}$$

$$\dim_p(\mu) := \inf\{\dim_p(E) \mid \mu(E^c)=0\}.$$

Thm: Let μ be a Borel probability measure in $\mathbb{R}^d$. Then

i.) $\dim_{\mathcal{H}}(\mu) = \inf\left\{\alpha \;\middle|\; \begin{array}{l}\underline{D}_\mu(x) \leq \alpha \\ \text{for } \mu\text{-almost all } x\end{array}\right\}.$

$$= \operatorname{ess\,sup}_\mu \underline{D}_\mu(x)$$

where $\underline{D}_\mu(x) = \liminf_{n\to\infty} \dfrac{\log(B(x,r_n))}{\log(r_n)}.$

ii.) $\dim_p(\mu) = \operatorname{ess\,sup}_\mu \overline{D}_\mu(x)$

iii.) If $\mu(E)=1$ and $\underline{D}_\mu(x)=\alpha$ $\forall x\in E$, then $\dim_{\mathcal{H}}(E)=\alpha$

iv.) If $\mu(E)=1$ and $\overline{D}_\mu(x)=\alpha$ $\forall x\in E$, then $\dim_p(E)=\alpha$.

The proof is essentially Frostman + mass distribution.