The next question that we would like to explore is the following:

<u>Q:</u> Suppose $\dim_{\mathcal{H}}(\Delta) < \text{sim-dim}(\Phi)$ what can we say about

$$\{\phi_{\underline{i}}^{-1} \circ \phi_{\underline{j}} \mid \underline{i}, \underline{j} \in \Sigma^*\}?$$

The Bandt-Graf-Schief thm allows us to argue that if $s = \text{sim-dim}(\Phi)$, and $\dim_{\mathcal{H}}(\Delta) < s$, then

$$\mathcal{H}^s(\Delta) = 0$$

Thus condition BG1 does not hold.

In other words,

$$Id \in \overline{\{\phi_{\underline{i}}^{-1} \circ \phi_{\underline{j}}\}}$$

or $\exists \{\underline{i}_k\}_{k=1}^{\infty} \{\underline{j}_k\}_{k=1}^{\infty}$ s.t. $\phi_{\underline{i}_k}^{-1} \circ \phi_{\underline{j}_k} \xrightarrow{k \to \infty} Id.$

Conjecture:

Suppose Φ is an IFS on $\mathbb{R}^d$.

If $\dim_{\mathcal{H}}(\Lambda) < \text{sim-dim}(\Phi)$

then $\exists\, \underline{i}, \underline{j} \in \Sigma^*$ s.t. $\underline{i} \neq \underline{j}$ and

$$\phi_{\underline{i}} = \phi_{\underline{j}} \quad \text{(with caveats)}$$

The Caveats

For $d \geq 2$, this cannot possible hold if one considers <u>any</u> IFS.

For example,

let $\Phi = (\phi_i)_{i=1}^{8}$

$$\phi_1(x) = \tfrac{1}{3}x \qquad \phi_5(x) = \tfrac{1}{3}x + (\tfrac{2}{3}, 0)$$

$$\phi_2(x) = \tfrac{1}{3}x + (0, \tfrac{1}{3}) \qquad \phi_6(x) = \tfrac{1}{3}x + (\tfrac{2}{3}, 1/3)$$

$$\phi_3(x) = \tfrac{1}{3}x + (0, 2/3) \qquad \phi_7(x) = \tfrac{1}{3}x + (\tfrac{2}{3}, 2/3)$$

$$\phi_4(x) = \tfrac{1}{3}x + (0, t) \qquad \phi_8(x) = \tfrac{1}{3}x + (\tfrac{2}{3}, t)$$

where $t = 2\sum_{k=1}^{\infty} 3^{-k^2}$

Then $\text{sim-dim}(\Phi) = \dfrac{\log(8)}{\log(3)}$

IF Λ is the attractor, then

$$\dim_{\mathcal{H}}(\text{proj}_1(\Lambda)) = \dim_p(\text{proj}_1(\Lambda)) = \frac{\log(2)}{\log(3)}$$

and $\dim_{\mathcal{H}}(\text{proj}_2(\Lambda)) = \dim_p(\text{proj}_2(\Lambda)) = 1$

$$\Rightarrow \quad \dim_{\mathcal{H}}(\Lambda) \le 1 + \frac{\log(2)}{\log(3)}$$

Therefore,

$$\dim_{\mathcal{H}}(\Lambda) \le 1 + \frac{\log(2)}{\log(3)} < 1 + \frac{\log(8/3)}{\log(3)} = \frac{\log(8)}{\log(3)}$$

$$= \text{sim-dim}(\Phi)$$

$\phi_{\underline{i}} = \phi_{\underline{j}}$ and $\ell(\underline{i}) = \ell(\underline{j}) = n$

$\Leftrightarrow \quad ct = d$

where $c = \sum_{k=1}^{n} \varepsilon_k 3^{-k}$ $\quad d = \sum_{k=1}^{n} \hat{\varepsilon}_k 3^{-k}$

$\varepsilon_k = 1$ or 0 $\quad \hat{\varepsilon}_k = 0, 1$ or 2

Since $t = 2\sum_{k=1}^{\infty} 3^{-k_2}$

this is impossible.

So for $d \geq 2$, we need to assume that if $\phi_i(x) = r_i A_i x + x_i$ then we need to assume

$\nexists$ hyperplane, V, s.t.

$$A_i V = V \text{ for all } i$$

The dimension one case is what we will focus on to avoid these complications. Also, technically this conjecture remains unresolved.

What is known in the 1-d case

- Hochman ('14), If $\phi_j(x) = r_j x + x_j$ $r_j \in (-1,1) \setminus \{0\}$ and $x_j \in \mathbb{R}$ are algebraic, then either
  - $\dim_H(\Delta) = \min(\text{sim-dim}(\Phi), 1)$

  or
  - $\exists \underline{i}, \underline{j} \in \Sigma_1^*$ s.t. $\phi_{\underline{i}} = \phi_{\underline{j}}$ and $\underline{i} \neq \underline{j}$

- Rapaport, Varjú ('20)

  If $x_j \in \mathbb{Q}$, then

  $\dim_H(\Lambda) < \min(\text{sim-dim}(\Phi), 1)$

  $\Rightarrow \exists\, \underline{i}, \underline{j} \in \Sigma^*$ s.t. $\phi_{\underline{i}} = \phi_{\underline{j}}$

- Rapaport ('22)

  If r_j is algebraic for all j, then

  $\dim_H(\Lambda) < \min(\text{sim-dim}(\Phi), 1)$

  $\Rightarrow \exists\, \underline{i}, \underline{j} \in \Sigma^*$ s.t. $\phi_{\underline{i}} = \phi_{\underline{j}}$

Hochman's Result

Let Φ be a IFS on $\mathbb{R}$

For $n \in \mathbb{Z}_+$, we define

$$\Delta_n := \min\left\{ \operatorname{dist}(\Lambda_{\underline{i}}, \Lambda_{\underline{j}}) \;\middle|\; \underline{i} \neq \underline{j},\ \underline{i}, \underline{j} \in \Sigma_n \right\}.$$

Thm* (Hochman '14)

Let Φ be an IFS on $\mathbb{R}$.

If $\dim_H(\Lambda) < \min(\text{sim-dim}(\Phi), 1)$

then $-\frac{1}{n}\log(\Delta_n) \xrightarrow{n\to\infty} \infty$

(i.e. $\Delta_n \to 0$ super-exponentially).

Corollary to Thm *

If $\phi_i(x) = r_i x + x_i$ and r_i and x_i are algebraic then

$$\dim_{\mathcal{H}}(\Lambda) < \min(\text{sim-dim}(\Phi), 1)$$
$$\Rightarrow \phi_{\underline{i}} = \phi_{\underline{j}} \text{ for some } \underline{i} \neq \underline{j}$$

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pf of corollary

For each $\underline{i}, \underline{j} \in \Sigma_n$,

$$|\phi_{\underline{i}}(0) - \phi_{\underline{j}}(0)|$$

is a polynomial in $\{r_j\}_{j=1}^m$ and $\{x_j\}_{j=1}^m$

Since all of the r_j and x_j are algebraic, there exists $s = s(r_j, x_j)$ s.t. either

$$|\phi_{\underline{i}}(0) - \phi_{\underline{j}}(0)| = 0 \quad \text{or}$$

$$|\phi_{\underline{i}}(0) - \phi_{\underline{j}}(0)| > s^n.$$

If $\dim_{\mathcal{H}}(\Lambda) < \min(\text{sim-dim}(\Phi), 1)$ then the second option is not possible.

$\square$.

Another Corollary of Hochman's result

Recall that we used transversality and Riesz energies to prove that if $E \subset \mathbb{R}^2$ and $\dim_{\mathcal{H}}(E) = s$ then for almost every $\ell \in G(2,1)$

$\dim_{\mathcal{H}}(\mathrm{proj}_\ell(E)) = s.$

Hochman's thm allows us to deduce a much stronger estimate on the size of the set of projections for which $\dim_{\mathcal{H}}(\mathrm{proj}_\ell(E)) < s$ when E is an attractor of a nice IFS.

Corollary: Consider $\Phi = (\phi_i)_{i=1}^4$ the IFS corresponding to the 4-corner Cantor set. Consider the parametrized family of linear maps

$$P_t : \mathbb{R}^2 \to \mathbb{R}, \quad P_t(x,y) = tx + y.$$

Let $\Lambda_t = P_t(\Lambda)$.

If $t \notin \mathbb{Q}$, then $\dim_{\mathcal{H}}(\Lambda) = 1$.

Pf: For each $t \in \mathbb{R}$,

Λ_t is the attractor of the IFS

$$\Phi_t = \left\{ \phi_i^t(x) = \frac{1}{4}x + P_t x_i \;\middle|\; i = 1, \dots, 4. \right\}.$$

$$\phi_1^t(x) = \frac{1}{4}x, \quad \phi_2^t(x) = \frac{1}{4}x + \frac{3}{4}$$

$$\phi_3^t(x) = \frac{1}{4}x + \frac{3}{4}t \qquad \phi_4^t(x) = \frac{1}{4}x + \frac{3}{4} + \frac{3}{4}t.$$

Fix $t \in \mathbb{R}$ and suppose $\dim_{\mathcal{H}}(\Lambda_t) < 1$.

Note that $\text{sim-dim}(\Phi_t) = 1$, so Hochman's thm applies.

For $n \in \mathbb{Z}_+$, let

$$\varepsilon = (0, 3, 3+3t, 3t)$$

then for $\underline{i} \in \Sigma_n = \{1, \dots 4\}^n$.

$$\phi_{\underline{i}}(0) = \sum_{k=1}^{n} \varepsilon_{i_k} 4^{-k}$$

Thus for $\underline{i}, \underline{j} \in \Sigma_n$,

$$\operatorname{dist}(\Lambda^t_{\underline{i}}, \Lambda^t_{\underline{j}}) = |\phi_{\underline{i}}(0) - \phi_{\underline{j}}(0)|$$

$$= p_{\underline{i},\underline{j}} - t \cdot q_{\underline{i},\underline{j}}$$

where

$$p_{\underline{i},\underline{j}}, q_{\underline{i},\underline{j}} \in \left\{ \sum_{k=1}^{n} a_k 4^{-k} \,\middle|\, a_k \in \{0,1,2,3\} \right\}.$$

Hochman's thm implies that for m large enough, $\exists\, p_m, q_m$ s.t.

$$p_m, q_m \in \left\{ \sum_{k=1}^{m} a_k 4^{-k} \,\middle|\, a_k \in \{\pm 3, 0\} \right\}.$$

and

$$|p_m - t q_m| < 4^{-1000m}$$

If $q_m = 0$, then

$$|p_m| < 4^{-1000m}$$

but since $p_m = \sum_{k=1}^{m} a_k 4^{-k}$,

$$|p_m| \geq 4^{-m} \text{ if } p_m \neq 0,$$

thus

$$p_m = 0$$

and $|p_m - t q_m| = 0$.

If $q_m \neq 0$ for all m large enough, then

$$\left| t - \frac{p_m}{q_m} \right| < 4^{-500m}$$

$$\Rightarrow \left| \frac{p_{m+1}}{q_{m+1}} - \frac{p_m}{q_m} \right| < 4^{-100m}$$

But $\frac{p_{m+1}}{q_{m+1}} - \frac{p_m}{q_m} = \left\{ \sum_{k=-2m}^{2m} b_k 4^{-k} \,\middle|\, b_k \in \{0,1,2,3\} \right\}$

$$\Rightarrow \quad \frac{p_{m+1}}{q_{m+1}} - \frac{p_m}{q_m} = 0 \quad \text{or}$$

$$\left| \frac{p_{m+1}}{q_{m+1}} - \frac{p_m}{q_m} \right| \geq 4^{-2m}$$

Thus

$$\frac{p_{m+1}}{q_{m+1}} = \frac{p_m}{q_m} \quad \text{for all } m \text{ large}$$

enough

$$\Rightarrow \quad \frac{p_m}{q_m} \xrightarrow{m \to \infty} t$$

where $\frac{p_m}{q_m} = \frac{p_1}{q_1} \quad \forall m$

$$\Rightarrow t = \frac{p_1}{q_1}$$

$$\Leftrightarrow \quad | p_1 - t q_1 | = 0 \qquad \square.$$