Thm (Csörnyei, Jordan, Policott, Preiss, Solomyak)

There exist a parametrized family of IFS, $\{\Phi_t\}_{t\in I}$ in $\mathbb{R}^2$ such that

- $\text{sim-dim}(\Phi_t) = \dfrac{\log(10)}{\log(3)}$ $\forall t$
- $\mathcal{L}^2(\Lambda_t) > 0$ for all t
- $\text{int}(\Lambda_t) = \phi$ for all t.

The Construction

Let $t = (t_1, t_2) \in [0,1]^2 \subset \mathbb{R}^2$

Then $\Phi_t = (\phi_i)_{i=}$

$T_1(x) = \frac{1}{3}x$

$T_2(x) = \frac{1}{3}x + (0, t_2)$

$T_3(x) = \frac{1}{3}x + (0, t_2)$

$T_4(x) = \frac{1}{3}x + (0, 1)$

$T_5(x) = \frac{1}{3}x + (\frac{1}{3}, 0)$

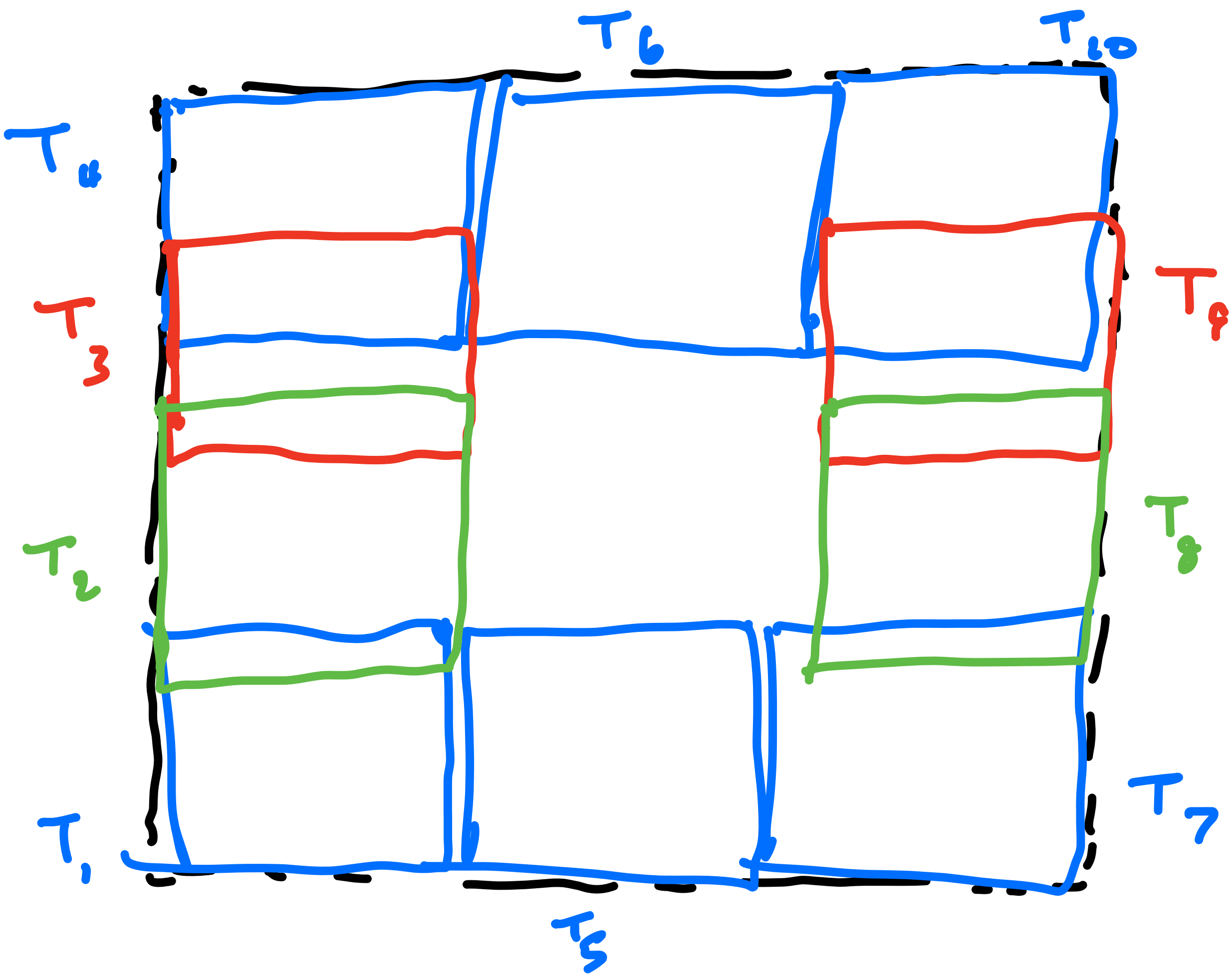
$T_6(x) = \frac{1}{3}x + (\frac{1}{3}, 1)$

$T_7(x) = \frac{1}{3}x + (\frac{2}{3}, 0)$

$T_8(x) = \frac{1}{3}x + (\frac{2}{3}, t_2)$

$T_9(x) = \frac{1}{3}x + (\frac{2}{3}, t_2)$

$T_{10}(x) = \frac{1}{3}x + (\frac{2}{3}, 1)$



Observe that for lines of the form

$\{(k+\frac{1}{2})3^{-n}\} \times \mathbb{R}$, with $n \geq 0$,

$0 \leq k \leq 3^n - 1$. Then intersection

of Λ_t and $\{(k+\frac{1}{2})3^{-n}\} \times \mathbb{R}$

has zero $\mathcal{L}^1$-measure

Thus Λ_t has empty interior for all $t \in [0,1]^2$.

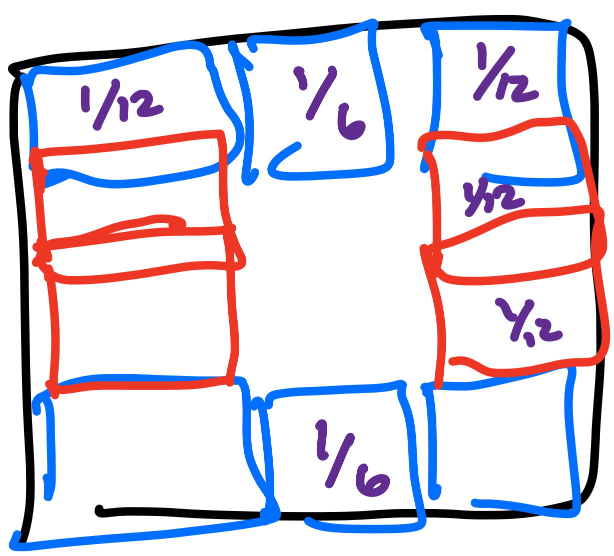
For each $t \in [0,1]^2$, let

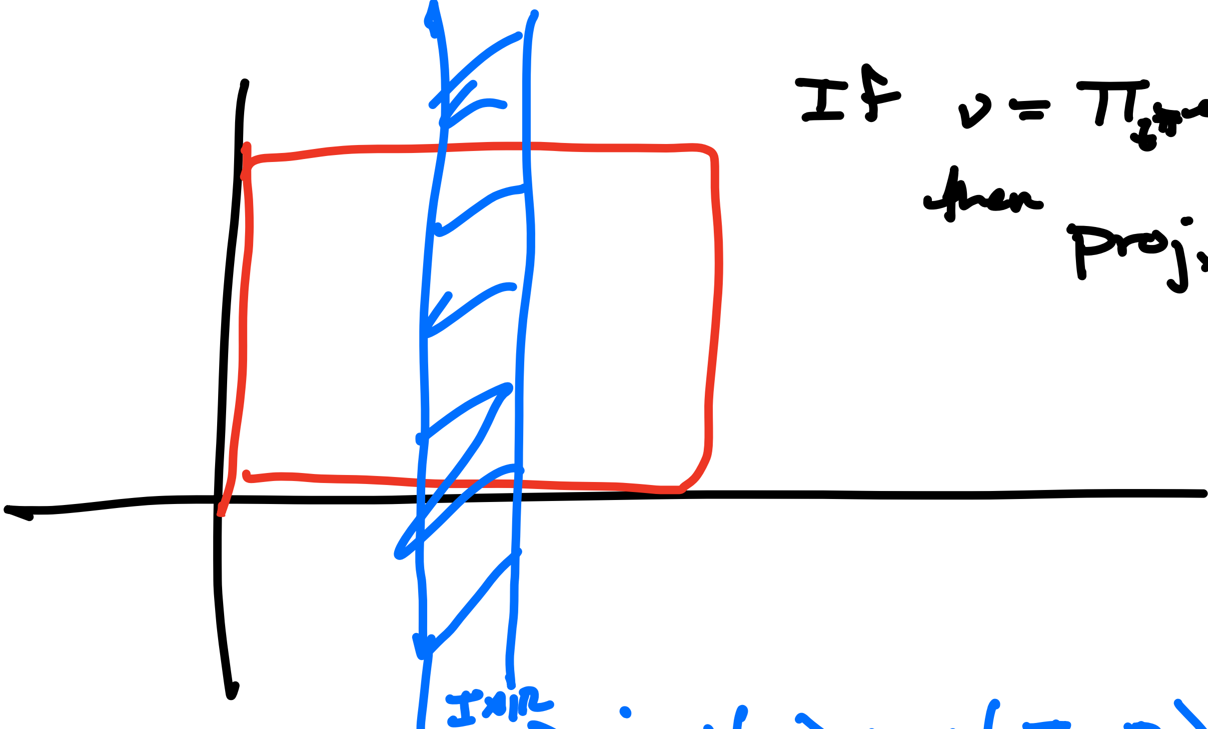
$$\pi_t : \Sigma_1 \to \Lambda_t$$ be the natural projection.

Define a measure, μ on Σ_1 by

$$\mu([j]) = \frac{1}{12} \quad \text{for } j = 1, \dots, 4, \; 7, \dots, 10$$

$$\mu([j]) = \frac{1}{6} \quad \text{for } j = 5, 6$$





If $\nu = \Pi_{t*}\mu$

then $\text{proj}_{x_1} \nu = \text{Leb.}$

$\text{proj}_{x_1} \nu(I) = \nu(I \times \mathbb{R})$.

It suffices to show that for a.e. x_1, $\nu\big|_{\{x_1\}\times\mathbb{R}} =: \nu_{t,x_1}$ is absolutely continuous w.r.t. Lebesgue measure.

Let $\tilde{\Sigma}_1' = \{1,2,3\}^{\mathbb{N}}$

Define the map $p : \Sigma_1' \to \tilde{\Sigma}_1'$ by

$p(\underline{i})_n = g(i_n)$

where

$$g(j) = \begin{cases} 0 & j = 1,2,3,4 \\ 1 & j = 5,6 \\ 2 & j = 7,8,9,10. \end{cases}$$

$$P_\# \mu =: \bar{\mu} = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)^{\mathbb{N}}.$$

Given $\underline{\xi} \in \tilde{\Sigma}_1$, let

$$\mu_{\underline{\xi}} = \text{induced measure on } p^{-1}(\underline{\xi}).$$

For $\underline{k} \in \Sigma_N^1$ $\mu_{\underline{\xi}}([\underline{k}] \cap p^{-1}(\underline{\xi})) = \dfrac{\mu([\underline{k}])}{P_\# \mu([p\,\underline{k}])}$
$= \dfrac{\mu([\underline{k}])}{3^{-N}}$

Then

$$\pi_{t,\underline{\xi}} : p^{-1}(\underline{\xi}) \to \{x_1\} \times \mathbb{R}$$

and $\pi_{t,\underline{\xi}}\, \mu_{\underline{\xi}} = \nu_{t, x_1}$

where $\tilde{\pi} : \tilde{\Sigma}_1 \to [0,1)$ and $\tilde{\pi}(\underline{\xi}) = x_1$.

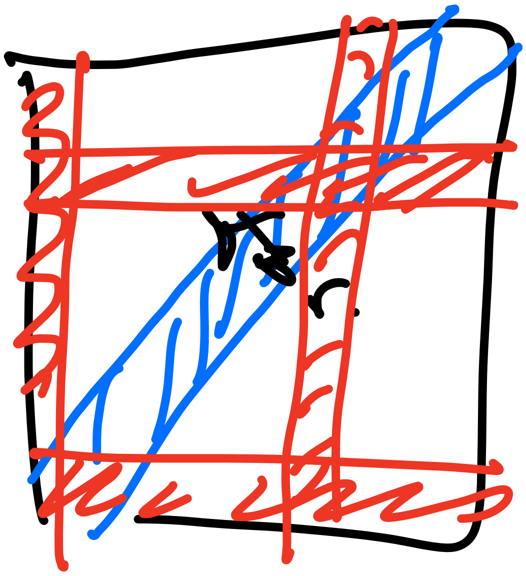
Note that $\underline{j_1}, \underline{j_2} \in p^{-1}(\underline{\xi})$ with

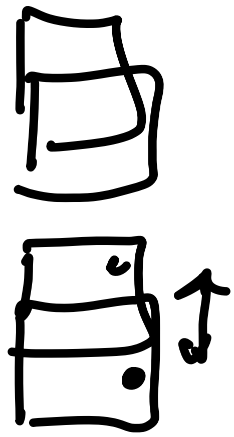
$$|\underline{j_1} \wedge \underline{j_2}| = n$$

Then

$$\left| \pi_{t,\underline{\xi}}(\underline{j_1}) - \pi_{t,\underline{\xi}}(\underline{j_2}) \right| \geq 3^{-n} \min \begin{pmatrix} |t_1|, |t_2| \\ |t_1 - \tfrac{2}{3}|, |t_2 - \tfrac{2}{3}| \\ |t_1 - t_2| \end{pmatrix}$$

$\boxed{(\text{where } t = (t_1, t_2))}$





can always make these seperate.

$$\mathcal{L}^2\left(\left\{t\in[0,1]^2 \;\middle|\; |\Pi_{t,\underline{\xi}}(j_1)-\Pi_{t,\underline{\xi}}(j_2)|\le r\right\}\right)$$

$$\lesssim_n 3^n r.$$

Idea of Mattila

Now, our goal is to show that $\int_{\{x_2\}\times\mathbb{R}} D(\nu_{t,x_1})^2 < \infty$ for a.e. $t\in[0,1]^2$

To that end,

$$\int_{[0,1]^2}\left(\int_{\{x_2\}\times\mathbb{R}} D(\nu_{t,x_2})(y)\, d\nu_{t,x_2}(y)\right) dt$$

$$= \int_{[0,1]^2}\int \liminf_{r\to 0}\frac{\nu_{t,x_2}([y-r,y+r])}{2r}\, d\nu_{t,x_2}\, dt.$$

Fatou

$$\leq \liminf_{r \to 0} \frac{1}{2r} \int_{[0,1]^2} \int_{\{x_1\} \times \mathbb{R}} \nu_{t,x_2}([y-r, y+r]) \, d\nu_{t,x_2} \, dt$$

$$= \liminf_{r \to 0} \frac{1}{2r} \int \int \int \chi_{[y-r, y+r]}(z) \, d\mu_{\underline{\xi}} \, d\mu_{\underline{\xi}}$$

$$\leq \liminf_{r \to 0} \frac{1}{2r} \int_{[0,1]^2} \int_{(p^{-1}(\underline{\xi}))^2} \chi_{\{|\pi_{t,x_2}(\underline{i}) - \pi(\underline{j})| < r\}}$$

$$= \liminf_{r \to 0} \frac{1}{2r} \int_{(p^{-1}(\underline{\xi}))^2} \mathcal{L}^2(\{|\pi_{t,x_2}(\underline{i}) - \pi_{t,x_2}(\underline{j})| < r\}) \, d\mu_{\underline{\xi}}(\underline{i}) \, d\mu_{\underline{\xi}}(\underline{j})$$

$$\leq \liminf_{r \to 0} \frac{1}{2r} \int_{(p^{-1}(\underline{\xi}))^2} 3^{|\underline{i} \wedge \underline{j}|} r \, d\mu_{\underline{\xi}}(\underline{i}) \, d\mu_{\underline{\xi}}(\underline{j})$$

For fixed $\underline{j}$, let

$$\pi_n(\underline{j}) = (j_1, \ldots, j_n)$$

Then

$$3^{|\underline{i} \wedge \underline{j}|} = \sum_{n=1}^{\infty} 3^n \chi_{E_n(\underline{j})}$$

where

$$E_n(\underline{j}) = \left\{ \underline{i} \in \Sigma \;\middle|\; \begin{array}{l} \pi_n(\underline{i}) = \pi_n(\underline{j}) \text{ but} \\ \pi_{n+1}(\underline{i}) \neq \pi_{n+1}(\underline{j}). \end{array} \right\}$$

$$= [\pi_n(\underline{j})] \setminus [\pi_{n+1}(\underline{j})]$$

For any measure, λ on Σ, and $F \subset \Sigma$,

$$\int_F 3^{|\underline{i} \wedge \underline{j}|} \, d\lambda(\underline{i})$$

$$= \sum_{n=1}^{\infty} 3^n \lambda\big(F \cap [\pi_n(\underline{j})] \setminus [\pi_{n+1}(\underline{j})]\big)$$

$$\leq \sum_{n=1}^{\infty} 3^n \lambda(F \cap [\pi_n(\underline{j})])$$

Now for $G \in \Sigma_1$

$$\int_G \int_F 3^{|\underline{i} \wedge \underline{j}|} \, d\lambda(\underline{i}) \, d\lambda(\underline{j})$$

$$\leq \int_G \sum_{n=1}^{\infty} 3^n \lambda(F \cap [\pi_n(\underline{j})] \, d\lambda(\underline{j})$$

$$= \sum_{n=1}^{\infty} 3^n \int_G \lambda(F \cap [\pi_n(\underline{j})]) \, d\lambda(\underline{j})$$

$$= \sum_{n=1}^{\infty} 3^n \sum_{\underline{k} \in \Sigma_n^1} \lambda(F \cap [\underline{k}]) \lambda(G \cap [\underline{k}])$$

Thus

$$\int_{p^{-1}(\xi)} \int_{p^{-1}(\xi)} 3^{|\underline{i} \wedge \underline{j}|} \, d\mu_{\underline{\xi}}(\underline{i}) \, d\mu_{\underline{\xi}}(\underline{j})$$

$$\leq \sum_{n=1}^{\infty} 3^n \sum_{\underline{k} \in \Sigma_n^1} \mu_{\underline{\xi}}(p^{-1}(\underline{\xi}) \cap [\underline{k}])^2$$

Estimating $\mu_{\underline{\xi}}(p^{-1}(\underline{\xi}) \cap [\underline{k}])$

Let $k \in \{1, \dots, 10\}$

and suppose $\pi_2(\xi) = 1$

$\Rightarrow$ $[k] \cap p^{-1}(\xi) \neq \phi \Rightarrow p(k) = \pi_2(\xi) = 1$

$\Rightarrow$ $k = 1, 2, 3$ or 4.

Now $\mu([k]) = \frac{1}{12}$ and

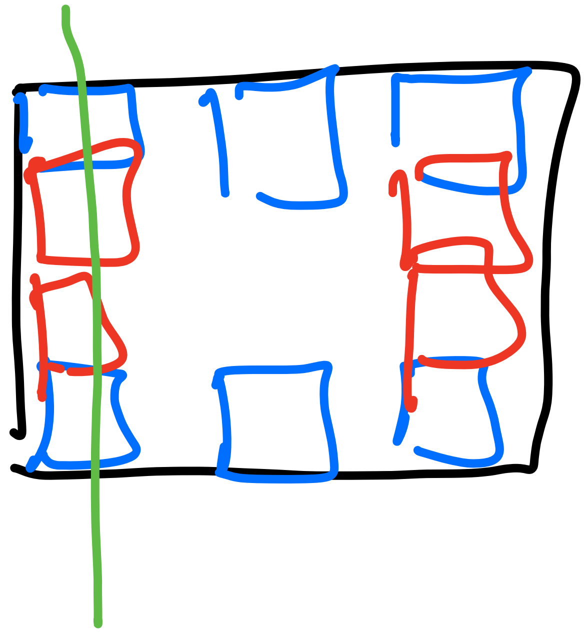
$$\mu_{\underline{\xi}}([k] \cap p^{-1}(\xi)) = \frac{\mu([k])}{p_*\mu([p(k)])}$$

$$= 3 \cdot \frac{1}{12} = 1/4.$$

Similarly, if $\pi_2(\xi) = 2$,

then $\mu_{\underline{\xi}}([k] \cap p^{-1}(\xi)) = \frac{1}{2}.$

and if $\pi_2(\xi) = 3$, $\mu_\xi([k] \cap p^{-1}(\xi)) = 1/4.$



For $\underline{k} \in \Sigma_N$

$$\mu_{\underline{\xi}}([\underline{k}] \cap p^{-1}(\xi)) = 4^{-n_1} \, 2^{-n_2} \, 4^{-n_3}$$

where $n_1(M) = \#\{n \leq M \mid \pi_n(\xi) = 1\}$

$n_2(M) = \#\{n \leq M \mid \pi_n(\xi) = 2\}$.

$n_3(M) = \#\{n \leq M \mid \pi_n(\xi) = 3\}$.

<u>Claim:</u> For a.e. $\underline{\xi}$, $\exists N$ s.t. if $M > N$, $n_1, n_2, n_3 > (\frac{1}{3} - \delta) M$

The claim implies that for typical

$\underline{\xi}$ and for N large enough, if

$\underline{k} \in \Sigma'_N$ then

$$\mu_{\underline{\xi}}\left([\underline{k}] \cap p^{-1}(\xi)\right) \leq 4^{-\left(\frac{1}{3}-\delta\right)N} 2^{-\left(\frac{1}{3}-\delta\right)N} 4^{-\left(\frac{1}{3}-\delta\right)N}$$

$$= 4^{-\left(2/3-2\delta\right)N} 2^{-\left(\frac{1}{3}-\delta\right)N}.$$

Finally note that

$$\#\left\{\underline{k} \in \Sigma'_N \,\middle|\, [\underline{k}] \cap p^{-1}(\xi)\right\}$$

$$\leq 4^{n_1} 2^{n_2} 4^{n_3}.$$

Therefore

$$\sum_{n=1}^{\infty} 3^n \sum_{\underline{k} \in \Sigma'_n} \mu_{\underline{\xi}}\left([\underline{k}] \cap p^{-1}(\underline{\xi})\right)^2$$

$$\leq \sum_{n=1}^{\infty} 3^n \; 4^{-\left(\frac{2}{3}-2\delta\right)n} 2^{-\left(\frac{1}{3}-\delta\right)n}.$$

$$= \sum_{n=1}^{\infty} \left(3\, 4^{-\left(\frac{2}{3}-2\delta\right)} 2^{-\left(\frac{1}{3}-\delta\right)}\right)^n.$$

For δ small enough $< \infty$ $\left(\text{Since } 4^{-2/3} 2^{-1/3} = (32)^{-1/3} < \underbrace{(27)^{-1/3}}_{=3^{-1}}\right)$

Thus

$$\int_{[0,1]^2} \int_{\{x_2\}\times\mathbb{R}} \left(D\, \nu_{t,x_2} \right)^2 d\mathcal{L}^1 \, d\mathcal{L}^2 < \infty.$$

$$\Rightarrow \quad \nu_{t,x_2} << d\mathcal{L}^1$$

Since ν_{t,x_2} is a probability measure on $\{x_2\}\times\mathbb{R} \cap \Lambda_t$,

$\mathcal{L}^1\left(\{x_2\}\times\mathbb{R} \cap \Lambda_t\right) > 0$ for a.e. $x_2 \in [0,1]$.

Since $\mathcal{L}^1(\mathrm{proj}_x(\Lambda_t)) = 1$,

$$\mathcal{L}^2(\Lambda_t) > 0.$$