The Equivalence of Conditions BG1, BG2, and BG3

$\exists \varepsilon > 0$

BG1: $\{\phi_i^{-1} \circ \phi_j\} \cap B(Id, \varepsilon) = \emptyset$.

BG2: $\sup_{x \in \Lambda} \|\phi_i(x) - \phi_j(x)\| \geq \varepsilon \cdot \min(r_i, r_j)$.

The equivalence of BG1 and BG2 is straightforward:

$$\sup_{x \in \Lambda} \|\phi_i(x) - \phi_j(x)\| \geq \varepsilon \cdot \min(r_i, r_j)$$

$$\Leftrightarrow \quad \sup_{x \in \Lambda} \|\phi_i^{-1} \circ \phi_j(x) - x\| \geq \varepsilon \quad \text{and}$$

$$\sup_{x \in \Lambda} \|\phi_j^{-1} \circ \phi_i(x) - x\| \geq \varepsilon.$$

If Λ is not contained in a hyperplane, the $\exists \{x_0, \ldots, x_d\} \subset \Lambda$ in general position

$$\Leftrightarrow \quad \sup_{\{x_0, \ldots, x_d\}} \|\phi_j^{-1} \circ \phi_i(x_k) - x_k\| \geq \varepsilon$$

$$\Leftrightarrow \quad \phi_j^{-1} \circ \phi_i \notin B(Id, c\varepsilon)$$ for some constant c.

If Λ is contained in a lower-dimensional subspace and $\Lambda \subset V$, then, given a set of points $\{x_0, \dots, x_d\}$ in general position, $\sup_{\{x_0,\dots,x_d\}} \|\phi_{\underline{j}}^{-1} \circ \phi_{\underline{i}}(x_k) - x_k\| \gtrsim \varepsilon$ may not imply $\sup_{x \in \Lambda} \|\phi_{\underline{j}}^{-1} \circ \phi_{\underline{i}}(x) - x\| \gtrsim \varepsilon$.

However,

$$\phi_i = \mathrm{proj}_V \circ \phi_i \circ \mathrm{proj}_V$$

so we can redefine the IFS on V, then run the same argument in the lower dimensional space.

Now, recall that

BG3: $\displaystyle \sup_{\ell > 0} \sup_{r_{\underline{k}} \sim \ell} \#\left\{ \underline{j} \;\middle|\; \begin{array}{c} r_{\underline{j}} \sim r_{\underline{k}} \\ \Lambda_{\underline{j}} \cap N_{\ell}(\Lambda_{\underline{k}}) \neq \phi \end{array} \right\} < \infty.$

$$\sim \sup_{\ell > 0} \sup_{r_{\underline{k}} \sim \ell} \#\left\{ \underline{j} \;\middle|\; \begin{array}{c} r_{\underline{j}} \sim r_{\underline{k}} \\ \mathrm{dist}(\Lambda_{\underline{j}}, \Lambda_{\underline{k}}) < \varepsilon\ell \end{array} \right\} < \infty.$$

First, note that if BG3 holds, then for $j \neq \underline{k}$, $\Lambda_j \cap \Lambda_{\underline{k}} = \phi$

If not, then for $\underline{i} \in \Sigma_1^*$, then

$$\Lambda_{\underline{i}j} \cap \Lambda_{\underline{i}\underline{k}} \neq \phi \quad \text{which implies}$$

$$\#\left\{ \underline{\ell} \;\middle|\; \begin{array}{l} r_{\underline{\ell}} \sim r_{\underline{i}j} \\ \mathrm{dist}(\Lambda_{\underline{\ell}}, \Lambda_{\underline{i}j}) < \varepsilon\, r_{\underline{i}j} \end{array} \right\} \geq m^{|\underline{i}|}$$

$$\Rightarrow \quad \tilde{\gamma}_\varepsilon = \infty.$$

Similary if for every $\delta > 0$, $\exists$ $j \neq \underline{k}$ such that $\mathrm{dist}(\Lambda_j, \Lambda_{\underline{k}}) < \delta \min(r_j, r_{\underline{k}})$ then $\exists\, c = c(\Lambda, d)$ s.t.

$$\tilde{\gamma}_{c\delta} = \infty.$$

Thus $\sup_{x \in \Lambda} \| \phi_{\underline{i}}(x) - \phi_{\underline{j}}(x) \| > \varepsilon$ for some $\varepsilon > 0$,

if $\tilde{\gamma}_\varepsilon < \infty$.

On the other hand, if BG1 holds,
fixing $\{x_0, \dots, x_d\} \subset \mathbb{R}^d$ in general position, we have for $\underline{r_i} \sim \underline{r_j}$, $i \neq j$

$$\sup_{l \in \{0,\dots,d\}} \| \phi_{\underline{i}}^{-1} \circ \phi_j(x_l) - x_l \| > \varepsilon .$$

Fix $\underline{k} \in \Sigma_1^{!*}$, and consider

$$\hat{\Gamma}_\varepsilon(\underline{k}) = \left\{ j \;\middle|\; \begin{array}{l} \underline{r_j} \sim r_{\underline{k}} \text{ and} \\ \operatorname{dist}(\Lambda_j, \Lambda_{\underline{k}}) < \varepsilon \min(r_j, r_{\underline{k}}) \end{array} \right\}$$

For every $j_1, j_2 \in \hat{\Gamma}_\varepsilon(\underline{k})$, $\exists\, l$ s.t.

$$\| \phi_{j_1}(x_l) - \phi_{j_2}(x_l) \| \geq \varepsilon r_{\underline{k}} .$$

$\exists\, L_d > 0$ s.t.

$$\# \left\{ j \in \hat{\Gamma}_\varepsilon(\underline{k}) \;\middle|\; \begin{array}{l} \text{s.t. for any } j_1, j_2 \text{ in this} \\ \| \phi_{j_1}(x_l) - \phi_{j_2}(x_l) \| \geq \varepsilon r_{\underline{k}} \end{array} \right\} \leq L_d$$

If $G = (\hat{\Gamma}_\varepsilon(\underline{k}), E)$ is a complete graph with nodes $\hat{\Gamma}_\varepsilon(\underline{k})$ and a coloring s.t. $e(j_1, j_2) = l$ s.t. $\| \phi_{j_1}(x_l) - \phi_{j_2}(x_l) \| \geq \varepsilon \underline{r_{\underline{k}}}$

Then G is a complete graph with cliques of size at most L_d.

Thus, a Ramsey theory-type counting argument yields a bound on the total nodes

$\Rightarrow$ $$\# \hat{\Gamma}_\varepsilon(\underline{k}) < C_d \text{ for all } \underline{k}$$

$$\Rightarrow \quad \hat{\gamma}_\varepsilon < \infty \quad .$$

Corollary:

Let Φ be an IFS on $\mathbb{R}^d$. Suppose Φ satisfies OSC and sim-dim$(\Phi) = d$. Then Λ has non-empty interior.

pf: Let $s = $ sim-dim(Φ)

By big thm, OSC $\Rightarrow$ $\mathcal{H}^d(\Lambda) = \mathcal{H}^s(\Lambda)$
$= \mathcal{L}^d(\Lambda) > 0$.

Let V be a nonempty bounded open set satisfying OSC. Then

$$\mathcal{L}^d\left(\bigcup_{i=1}^m \phi_i(V)\right) = \sum_{i=1}^m \mathcal{L}(\phi_i(V))$$
$$= \sum_{i=1}^m r_i^d \mathcal{L}^d(V) = \mathcal{L}(V).$$

$\Rightarrow$

$$\mathcal{L}^d\left(V \setminus \bigcup^m \phi_i(\bar{V})\right) \leq \mathcal{L}^d\left(V \setminus \bigcup \phi_i(V)\right) = 0.$$

$\Rightarrow$ $V \setminus \bigcup \phi_i(\bar{V})$ is an open set with zero Lebesgue measure, thus $V \setminus \bigcup_i \phi_i(\bar{V}) = \emptyset$.

Thus
$$\bar{V} = \bigcup_{i=1}^{m} \phi_i(\bar{V})$$
Since Λ is unique, $\bar{V} = \Lambda$

$\square$

Similarly, the Bandt, Graf, Schief theorem implies that if

- $\text{sim-dim}(\Phi) = d$
- $\mathcal{L}^d(\Lambda) > 0$

then Λ has nonempty interior. This leads to the following question:

Q: Suppose

- $\text{sim-dim}(\Phi) \geq d$
- $\mathcal{L}^d(\Lambda) > 0$

$\overset{?}{\Rightarrow}$ Λ has nonempty interior?

A: For $d \geq 2$, No

For $d = 1$, open question.

<u>Thm</u> (Csörnyei, Jordan, Policott, Preiss, Solomyak)

There exist a parametrized family of IFS, $\{\Phi_t\}_{t\in I}$ in $\mathbb{R}^2$ such that

- $\text{sim-dim}(\Phi_t) = \dfrac{\log(10)}{\log(3)} \quad \forall t$
- $\mathcal{L}^2(\Lambda_t) > 0$ for all t
- $\text{int}(\Lambda_t) = \emptyset$ for all t.

The Construction

Let $t = (t_1, t_2) \in [0,1]^2 \subset \mathbb{R}^2$

Then $\Phi_t = (\phi_i)_{i=}$

$T_1(x) = \frac{1}{3}x$

$T_2(x) = \frac{1}{3}x + (0, t_2)$

$T_3(x) = \frac{1}{3}x + (0, t_2)$

$T_4(x) = \frac{1}{3}x + (0, 1)$

$T_5(x) = \frac{1}{3}x + (\frac{1}{3}, 0)$

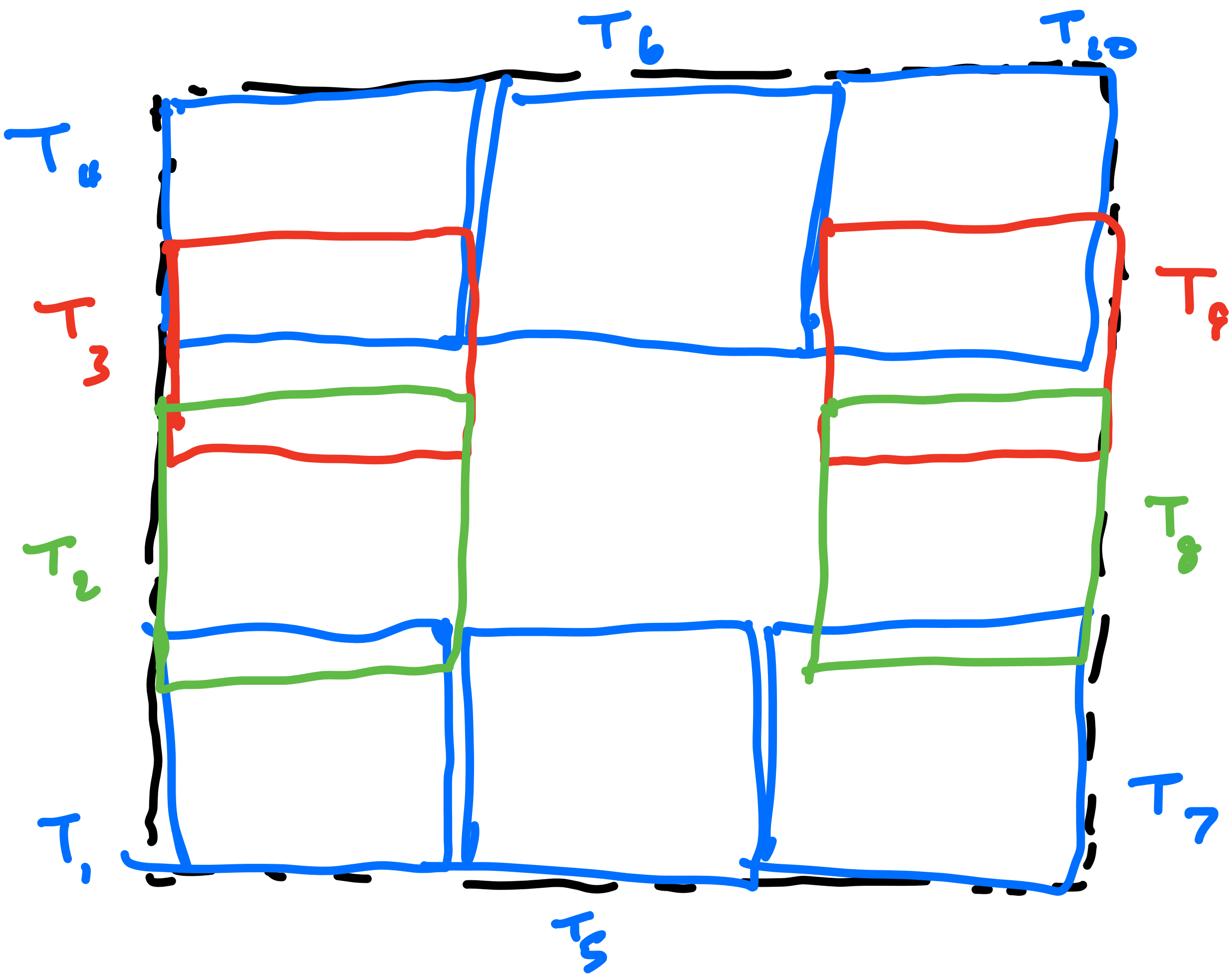
$T_6(x) = \frac{1}{3}x + (\frac{1}{3}, 1)$

$T_7(x) = \frac{1}{3}x + (\frac{2}{3}, 0)$

$T_8(x) = \frac{1}{3}x + (\frac{2}{3}, t_2)$

$T_9(x) = \frac{1}{3}x + (\frac{2}{3}, t_2)$

$T_{10}(x) = \frac{1}{3}x + (\frac{2}{3}, 1)$



Observe that for lines of the form

$\{(k+\frac{1}{2})3^{-n}\} \times \mathbb{R}$, with $n \geq 0$,

$0 \leq k \leq 3^n - 1$. Then intersection

of Λ_t and $\{(k+\frac{1}{2})3^{-n}\} \times \mathbb{R}$