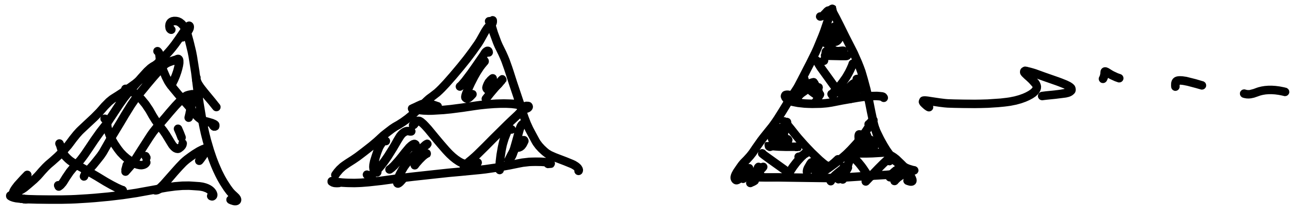
Introduction

The mathematical area of geometric measure theory is broad enough that naming a course "geometric measure theory" is not sufficient to give a clear picture of what will be covered. In this course, we will focus on two topics:

1. Fractals

2. Distances / Projections and other related problems including Kakeya.

The first topic concerns sets that are self-similar in nature. For example, sets that can be constructed iteratively, like the Sierpinski triangle



The second topic involves problems in analysis with discrete counterparts. For example, given N points in d-dimensional Euclidean space how many distinct distances must the set have? This is Erdös's distance ~~set~~ problem and it is unresolved in all dimensions. There is an equivalent analysis question: Given a set of dimension, s, must the set have a distinct distance set of positive Lebesgue measure.

Measure Theory Review

we will focus our attention on Euclidean space for now.

Outer Measures / Exterior Measures

Def: A function $\mu: \mathcal{P}(\mathbb{R}^d) \to [0,\infty]$ is called an outer measure on $\mathbb{R}^d$ provided that

i.) $\mu(\phi) = 0$

ii.) Countable subadditivity: if

$$A \subset \bigcup_{i=1}^{\infty} A_i \quad , \quad \text{then}$$

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

Def: A subset of the power set, $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^d)$, is a σ-algebra if

i.) $\phi, \mathbb{R}^d \in \mathcal{M}$

ii.) $\{A_i\}_{i=1}^{\infty} \subset \mathcal{M} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$

iii.) $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$.

Def: Given a σ-algebra, $\mathcal{M}$, a measure, $\mu : \mathcal{M} \to [0, \infty)$, is a function satisfying

i.) $\mu(\phi) = 0$

ii.) Countable additivity

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i) \quad , \mu(A_i \cap A_j) = 0$$

Def: A subset, $\mathcal{N} \subset \mathcal{P}(\mathbb{R}^d)$ is a <u>semi-algebra</u> if

i.) $\phi \in \mathcal{N}$

ii.) $A, B \in \mathcal{N} \Rightarrow A \cap B \in \mathcal{N}$

iii.) $A \in \mathcal{N} \Rightarrow \mathbb{R}^d \setminus A =$ finite union of pairwise disjoint elements of $\mathcal{N}$.

Def: A subset, $\mathcal{O} \subset \mathcal{P}(\mathbb{R}^d)$, is an <u>algebra</u> if

i.) $\phi \in \mathcal{O}$

ii.) $A, B \in \mathcal{O} \Rightarrow A \cap B \in \mathcal{O}$

iii.) $A \in \mathcal{O} \Rightarrow \mathbb{R}^d \setminus A \in \mathcal{O}$.

For any finite collection of sets, $\mathcal{F}$, let $\sigma(\mathcal{F})$ denote the smallest σ-algebra containing $\mathcal{F}$. Also known as the <u>σ-algebra generated by $\mathcal{F}$</u>.

<u>Thm:</u> (Measure Extension Thm)

Let $\mathcal{N}$ be a semi-algebra.

If $\mu : \mathcal{N} \to [0,\infty]$ is a σ-additive set-function, there is a unique measure defined on $\sigma(\mathcal{N})$ which extends μ.

Iterated Function Systems as a means of understanding dimension

<u>Def:</u> Let $m \geq 2$ and $d \geq 1$. We call the tuple $\Phi = (\phi_1, \ldots, \phi_m)$ of m contracting similarity transformations acting on $\mathbb{R}^d$ an IFS on $\mathbb{R}^d$ with contraction ratios $0 < r_i < 1$, $i = 1, \ldots, m$, if

$$\forall i \leq m, \quad \forall x, y \in \mathbb{R}^d,$$

$$\|\phi_i(x) - \phi_i(y)\| \leq r_i \|x - y\|.$$

We assume that not all of the fixed points of the maps in Φ are identical.

To every self-similar IFS, there exists a unique, compact set, Λ, satisfying

$$\Lambda = \bigcup_{i=1}^{m} \phi_i(\Lambda)$$

This is known as the <u>attractor</u> of the IFS

Examples

(1) Middle - Third Cantor Set

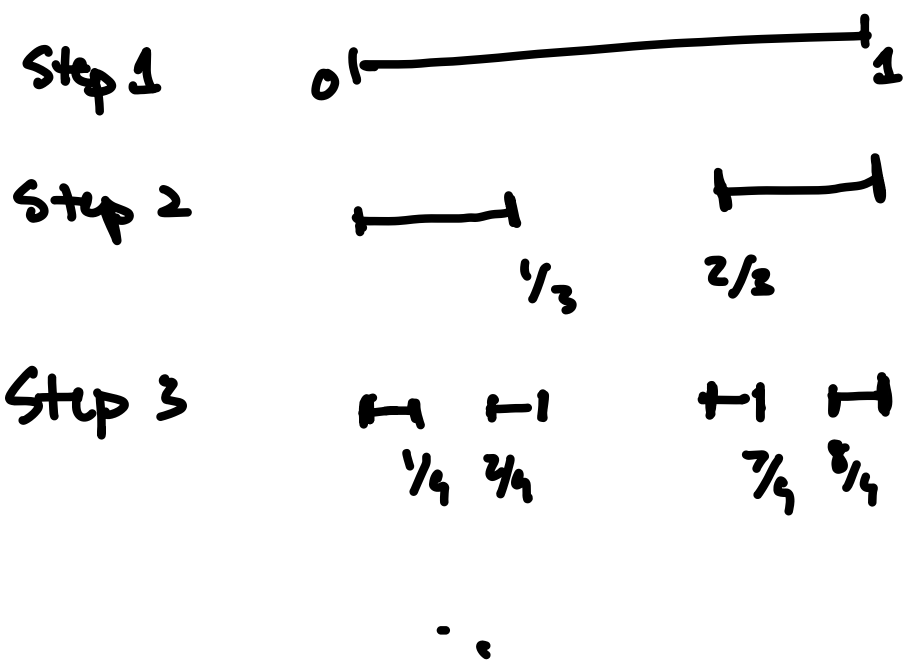
Define

$$\Phi_c := (\phi_1, \phi_2) \qquad \phi_j : \mathbb{R} \to \mathbb{R}.$$

$$\phi_1(x) = \frac{1}{3}x + 0$$

$$\phi_2(x) = \frac{1}{3}x + 2/3$$

We can visualize the attractor of this IFS in the following way:



② Sierpinski Carpet

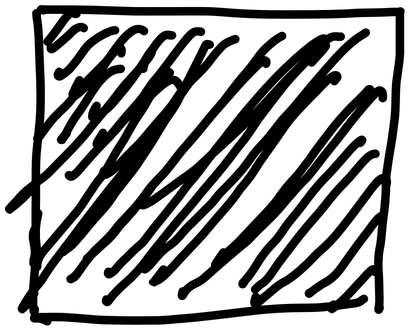
Define

$$\Phi_{sc} := (\phi_i)_{i=1}^{8}, \quad x_i \in \left\{ (0,0), (0,\tfrac{1}{3}), (\tfrac{1}{3},0), (0,\tfrac{2}{3}), (2/3,0), (\tfrac{1}{3},\tfrac{2}{3}), (\tfrac{2}{3},\tfrac{1}{3}), (\tfrac{2}{3},\tfrac{2}{3}) \right\}$$

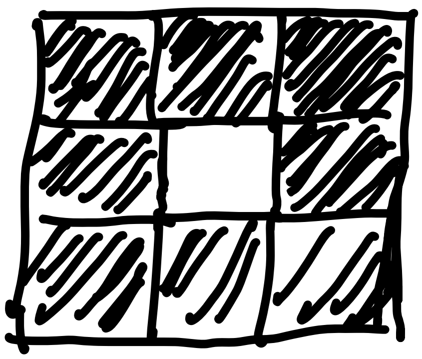
$$\phi_i : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\phi_i(x) = \frac{1}{3}x + x_i$$

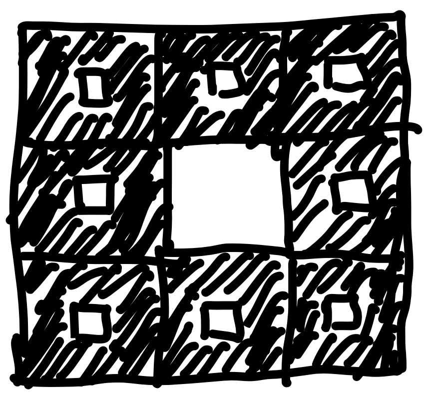
Step 1



Step 2



Step 3



(3) Line.

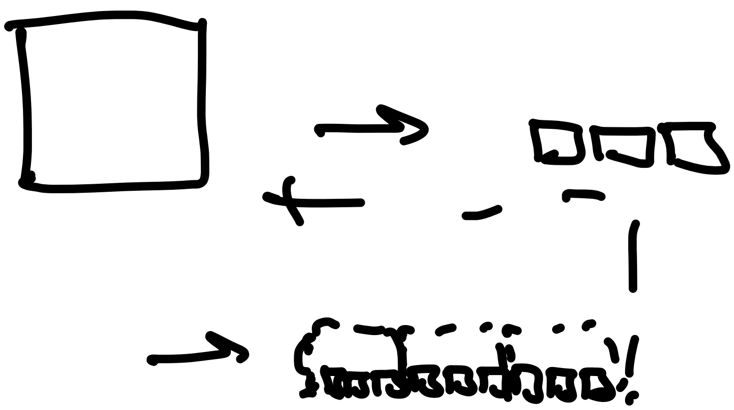
Define

$$\Phi_\ell := (\phi_i)_{i=1}^{3}$$

$$\phi_i : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\phi_1(x) = \tfrac{1}{3}x \qquad \phi_2(x) = \tfrac{1}{3}x + (\tfrac{1}{3}, 0)$$

$$\phi_3(x) = \tfrac{1}{3}x + (\tfrac{2}{3}, 0)$$



This example raises the question of what the best way to represent the sequence of sets that converge to the attractor.

Thm: Let $\Phi = (\phi_j)_{j=1}^{N}$ be an IFS on $\mathbb{R}^d$. Then there exists a unique, non-empty compact set $E \subset \mathbb{R}^d$ that satisfies

$$E = \bigcup_{j=1}^{N} \phi_j(E)$$

Define $$\Phi(A) := \bigcup_{j=1}^{N} \phi_j(A) =: \Phi^1(A)$$

and $$\Phi^{m+1}(A) := \Phi(\Phi^m(A)).$$

Then

1.) For any $A \subset \mathbb{R}^d$, A compact.

$$\lim_{m\to\infty} \Phi^m(A) = E$$ convergence in the Hausdorff metric.

2.) If $\Phi(A) \subset A$, then

$$E = \bigcap_{k=0}^{\infty} \Phi^k(A).$$

If $\mathcal{H}$ is the set of all compact sets then the Hausdorff metric on K is defined by

$$d(F_1, F_2) := \inf\{\delta \mid F_1 \subset N_\delta(F_2) \text{ and } F_2 \subset N_\delta(F_1)\}.$$

Pf: If A, B are compact, then

$$d(\Phi(A), \Phi(B)) = d\left(\bigcup_{j=1}^{N} \phi_j(A), \bigcup_{j=1}^{N} \phi_j(B) \right)$$

$$\leq \max_j d(\phi_j(A), \phi_j(B))$$

$$\leq \underbrace{\left(\max_j r_j\right)}_{<1} d(A,B)$$

Therefore, $\Phi : \mathbb{K} \to \mathbb{K}$ is a contraction mapping and has a unique fixed point and

$$\Phi^m(A) \xrightarrow{m\to\infty} E$$

Now for (2), if $\Phi(A) \subset A$.

$$\Phi^{k+1}(A) \subset \Phi^k(A)$$

$$\Rightarrow \bigcap_{k=0}^{M} \Phi^k(A) = \Phi^M(A) \xrightarrow{M\to\infty} E$$

$\square$.

Example:

Let $\Phi = (\phi_j)_{j=1}^N$ be an IFS on $\mathbb{R}^d$

with $\|\phi_j(x) - \phi_j(y)\| \leq r_j \|x-y\| < \|x-y\|$

and $b_j \in \mathbb{R}^d$ the unique points s.t.

$\phi_j(b_j) = b_j$.

Then for $R := \max_j \frac{\|b_j\|}{1-r_j}$

$$\Phi(\overline{B(0,R)}) \subset \overline{B(0,R)}$$

$$\Rightarrow \bigcap_{m=1}^{\infty} \Phi^m(\overline{B(0,R)}) = E = \text{attractor for } \Phi.$$
