Problem Set 2
Math 581A, Fall 2024

1. (Muscalu & Schlag, Problem 4.1) Compute the Fourier transform of the principal value of $1/x$. In other words, determine

$$\lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} e^{-2\pi i x \xi} \frac{dx}{x}$$

for every $\xi \in \mathbb{R}$. Conclude that (up to a normalizing constant) the Hilbert transform is an isometry on $L^2(\mathbb{R})$.

2. (Muscalu & Schlag, Problem 4.4) Prove that if $\operatorname{supp}(\hat{f}) \subset E \subset \mathbb{R}^d$, where E is measurable and f is Schwartz, say, prove that

$$\|f\|_q \leq |E|^{1/p-1/q} \|f\|_F \quad \forall 1 \leq p \leq q \leq \infty,$$

where $|E|$ stands for the Lebesgue measure of E. Hint: First handle the case $q = \infty$, $p = 2$ using Plancherel and Cauchy-Schwarz and then dualize and interpolate.

3. (Muscalu & Schlag, Exercise 7.8)
   (a) Show that $g \in \mathrm{BMO}([0,1])$ if and only if $a_n = O\ (1)$. In that case verify that $g(x) \simeq |\log x|$ for $0 < x < \frac{1}{2}$ (the function g is referred to as a discrete logarithm).
   (b) Show that $\chi_{[|x|<1]} \log |x| \in \mathrm{BMO}([-1,1])$ but
   $$\chi_{[|x|<1\}} \operatorname{sign}(x) \log |x| \notin \mathrm{BMO}([-1,1]).$$
   (c) Show that, for any f as in Definition 7.13,
   $$f^{\sharp\sharp}(x) := \sup_{x \in Q} \inf_{c} f_Q |f(y) - c| dy$$
   satisfies $f^{\sharp\sharp} \leq f^{\sharp} \leq 2 f^{\sharp\sharp}$.

4. (Muscalu & Schlag, Problem 10.1) Does there exist a nonzero function $f \in L^2\left(\mathbb{R}^d\right)$ such that we have both $f = 0$ and $f = 0$ on nonempty open sets?

5. (Muscalu & Schlag, Problem 10.2) Does there exist a nonzero $f \in L^2(\mathbb{R})$ with $f = 0$ on $[-1,1]$ and $f = 0$ on a half-line? Can we have $f = 0$ on a set of positive measure and $\hat{f} = 0$ on a half-line?

   Hint: Consider the circle case, and relate functions of this type to a suitable nonvanishing theorem in Chapter 3.

6. (Muscalu & Schlag, Problem 10.3) Suppose that $E, F \subset \mathbb{R}^d$ have finite measure. Show that for any $g_1, g_2 \in L^2\left(\mathbb{R}^d\right)$ there exists $f \in L^2\left(\mathbb{R}^d\right)$ with

$$f = g_1 \text{ on } E, \quad f = g_2 \text{ on } F.$$

   This can be seen as a statement that $f|_E$ and $\hat{f}\big|_F$ are independent.

7. (Muscalu & Schlag, Problem 10.4) Prove that, for $E, F \subset \mathbb{R}^d$ of finite measure, one has $\dim \left\{ f \in L^2\left(\mathbb{R}^d\right) \mid f = 0 \right.$ on $E, \hat{f} = 0$ on $\left. F \right\} = \infty$.