Problem Set 1
Math 581A, Fall 2024

1. (Muscalu & Schlag, Problem 1.1) Suppose that $f \in L^1(\mathbb{T})$ and that $\{S_n f\}_{n=1}^{\infty}$ (the sequence of partial sums of the Fourier series for f) converges in $L^p(\mathbb{T})$ to g for some $p \in [1, \infty]$ and some $g \in L^p(\mathbb{T})$. Prove that $f = g$. If $p = \infty$ conclude that f is continuous.

2. (Muscalu & Schlag, Exercise 1.2) Let $\mu \in \mathcal{M}(\mathbb{T})$ have the property that
$$\sum_{n \in \mathbb{Z}} |\hat{\mu}(n)| < \infty.$$
Show that $\mu(dx) = f(x)dx$, where $f \in C(\mathbb{T})$. Denote the space of all measure with this propeerty $\mathbb{A}(\mathbb{T})$ and identify there measures with their respective densities. Show that $\mathbb{A}(\mathbb{T})$ is an algebra under multiplication and that
$$\widehat{fg}(n) = \sum_{m \in \mathbb{Z}} \hat{f}(m)\hat{g}(n-m) \qquad \forall n \in \mathbb{Z},$$
where the sum on the right-hand side is absolutely convergent for every $n \in \mathbb{Z}$ and indeeed is absolutely convergent over all n. moreover, show that $\|fg\|_{\mathbb{A}} \leq \|f\|_{\mathbb{A}} \|g\|_{\mathbb{A}}$ where $\|f\|_{\mathbb{A}} := \|\hat{f}\|_{\ell^1}$. Finally, verify that if $f, g \in L^2(\mathbb{T})$ then $f * g \in \mathbb{A}(\mathbb{T})$.

3. (Muscalu & Schlag, Problem 1.5) Suppose that $\sum_{n=1}^{\infty} n|a_n|^2 < \infty$ and $\sum a_n$ is Cesàro summable. Show that $\sum a_n$ converges. Use this to prove that any $f \in C(\mathbb{T}) \cap H^{1/2}(\mathbb{T})$ satisfies $S_n f \to f$ uniformly.

4. (Muscalu & Schlag, Problem 1.8) Suppose that $f \in L^1(\mathbb{T})$ satisfies $\hat{f}(j) = 0$ for all j with $|j| < n$. Show that
$$\|f'\|_p \geq Cn\|f\|_p$$
for all $1 \leq p \leq \infty$, where C is independent of $n \in \mathbb{Z}^+$and of the choices of f and p.

5. (Muscalu & Schlag, Problem 1.11) Given N disjoint arcs $\{I_\alpha\}_{\alpha=1}^{N} \subset \mathbb{T}$, set $f = \sum_{\alpha=1}^{N} \chi_{I_\alpha}$. Show that
$$\sum_{|\nu|>k} |\hat{f}(v)|^2 \leq \frac{CN}{k}$$
Hint: The bound N^2/k is much easier and should be obtained first. Going from N^2 to N then requires one to exploit orthogonality in a suitable fashion.

6. (Muscalu & Schlag, Problem 1.12) Given any function $\psi : \mathbb{Z}^+ \to \mathbb{R}^+$such that $\psi(n) \to 0$ as $n \to \infty$, show that one can find a measurable set $E \subset \mathbb{T}$ for which
$$\limsup_{n \to \infty} \frac{|\widehat{\chi_E}(n)|}{\psi(n)} = \infty$$

7. (Muscalu & Schlag, Problem 1.13) In this problem the reader is asked to analyze some well-known partial differential equations in terms of Fourier series.

   (a) Solve the heat equation $u_t - u_{\theta\theta} = 0, u(0) = u_0$ (the data at time $t = 0$) on $\mathbb{T}$ using a Fourier series. Show that if $u_0 \in L^2(\mathbb{T})$ then $u(t, \theta)$ is analytic in θ for every $t > 0$ and solves the heat equation. Prove that $\|u(t) - u_0\|_2 \to 0$ as $t \to 0$. Write $u(t) = G_t * u_0$ and show that G_t is an approximate identity for $t > 0$. Conclude that $u(t) \to 0$ as $t \to 0$ in the L^p or $C(\mathbb{T})$ sense. Repeat for higher-dimensional tori.

   (b) Solve the Schrödinger equation $iu_t - u_{\theta\theta} = 0, u(0) = u_0$ on $\mathbb{T}$ with $u_0 \in L^2(\mathbb{T})$, using a Fourier series. In what sense does this Fourier series "solve" the equation? Show that $\|u(t)\|_2 = \|u_0\|_2$ for all t. Discuss the limit $u(t)$ as $t \to 0$. Repeat for higher-dimensional tori.

(c) Solve the wave equation $u_{tt} - u_{\theta\theta} = 0$ on $\mathbb{T}$ by Fourier series. Discuss the Cauchy problem as in (a) and (b). Show that if $u_t(0) = 0$ then, with $u(0) = f$,

$$u(t,\theta) = \frac{1}{2}(f(\theta+t) + f(\theta-t))$$

8. (Muscalu & Schlag, Problem 2.1) Let (X,μ) be a general measure space. We say a bounded sequence $\{f_n\}_{n=1}^\infty$ in $L^1(\mu)$ is uniformly integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable E one has

$$\mu(E) < \delta \quad \Longrightarrow \quad \sup_n \int_E |f_n|\,d\mu < \varepsilon$$

and there exists $E_0 \subset X$ with $\sup_{n\geq 1} \int_{X\setminus E_0} |f_n|\,d\mu < \varepsilon$. The same applies to any subset of L^1, not just to sequences. For simplicity, suppose now that μ is a finite measure.

(a) Let $\phi : [0,\infty) \to [0,\infty)$ be a continuous increasing function with $\lim_{t\to\infty} \phi(t)/t = +\infty$. Prove that

$$\sup_n \int \phi\left(|f_n(x)|\right)\mu(dx) < \infty$$

implies that $\{f_n\}$ is uniformly integrable. Conversely, show that this inequality is also necessary for uniform integrability and in particular that bounded sequences in $L^p(\mu)$ with $p > 1$ are uniformly integrable.

(b) Show that $\{f_n\}_{n=1}^\infty$ is uniformly integrable if and only if

$$\lim_{A\to\infty} \sup_{n\geq 1} \int_{[|f_n|>A]} |f_n(x)|\,\mu(dx) = 0$$

(c) Show that for an arbitrary sequence $\{f_n\}_{n=1}^\infty$ in $L^1(\mu)$ the following are equivalent: (i) $f_n \to f$ in $L^1(\mu)$ as $n \to \infty$; (ii) $f_n \to f$ in measure, with $\{f_n\}_{n\geq 1}$ uniformly integrable.

9. (Muscalu & Schlag, Problem 2.4) For any $f \in L^1\left(\mathbb{R}^d\right)$ and $1 \leq k \leq d$, let

$$M_k f(x) = \sup_{r>0} r^{-k} \int_{B(x,r)} |f(y)|dy.$$

Show that, for every $\lambda > 0$,

$$m_L\left(\{x \in L \mid M_k f(x) > \lambda\}\right) \leq \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R}^d)}$$

where L is an arbitrary affine k-dimensional subspace and m_L stands for Lebesgue measure (i.e., k-dimensional measure) on this subspace; C is an absolute constant.

10. (Muscalu & Schlag, Problem 2.5) Prove the Besicovitch covering lemma on $\mathbb{T}$. Suppose that $\{I_j\}$ are finitely many arcs with $|I_j| < 1$. Then there is a sub-collection $\{I_{j_k}\}$ such that the following properties hold:

(a) $\cup_k I_{j_k} = \cup_j I_j$

(b) No point belongs to more than C of the I_{j_k}, where C is an absolute constant. What is the optimal value of C ?

What can you say about the situation for higher dimensions (see for example Füredi and Loeb [44] and the references cited therein)?

11. (Muscalu & Schlag, Problem 2.12) Let $f \in L^1(\mathbb{T})$ satisfy $\hat{f}(n) = 0$ if $|n| > N$, where N is some positive integer. Show that there exists $E \subset \mathbb{T}$ with $|E| < \lambda^{-1}$ and with

$$\int_{\mathbb{T}\setminus E} \int_{\mathbb{T}} k_N(\theta)|f(\varphi - \theta)|^2 d\theta d\varphi \leq C\lambda \|f\|_1^2$$

where $k_N(\theta) := N\chi_{[2|\theta|N \leq 1]}$ is the box kernel and C is some absolute constant. Compare this with Exercise 2.11.