Pf:

- H and M are L^p-bounded for $p \in (1,\infty)$, thus.

$$\|H_*\|_{p\to p} \le \|MH\|_{p\to p} + \|M\|_{p\to p}.$$
$$\lesssim \|M\|_{p\to p}\left(\|H\|_{p\to p}+1\right) < \infty.$$

The weak-L^1 bound is the same as the weak-L^1 bound for H.

$\square$.

Corollary.

For any $f \in L^p$, $p \in [1,\infty)$

$$Hf(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{1}{x-y} f(y)\,dy.$$

exists almost everywhere.

Pf:

Define

$$\Lambda f(x) := \left| \limsup_{\varepsilon\to 0} H_\varepsilon f(x) - \liminf_{\varepsilon\to 0} (H_\varepsilon f)(x) \right|.$$

First, note that

$$\Lambda f \leq 2(H_* f)$$

For $p \in (1, \infty)$, let $\delta > 0$. Fix $f \in L^p$ and $g \in \mathcal{S}(\mathbb{R})$ s.t.

$$\|f-g\|_p$$

Since H_ε is linear and $g \in \mathcal{S}(\mathbb{R}$

$$\Lambda f(x) \leq \Lambda(f-g)(x).$$

Thus,

$$\|\Lambda f\|_p \leq \|\Lambda(f-g)\|_p \leq \delta.$$

$$\Rightarrow \quad \Lambda f = 0 \quad \checkmark$$

For $p = 1$, let $f \in L^1$ and $\delta > 0$, Let $g \in \mathcal{S}(\mathbb{R})$ satisfy $\|f-g\|_{L^1} < \delta^2$.

Then

$$|\{|\Lambda f| > \delta\}| \leq |\{|\Lambda(f-g)| > \delta\}|$$
$$\lesssim \frac{1}{\delta}\|f-g\|_{L^1}$$
$$\lesssim \delta.$$

$\square$

The L^∞-case.

We've already seen that

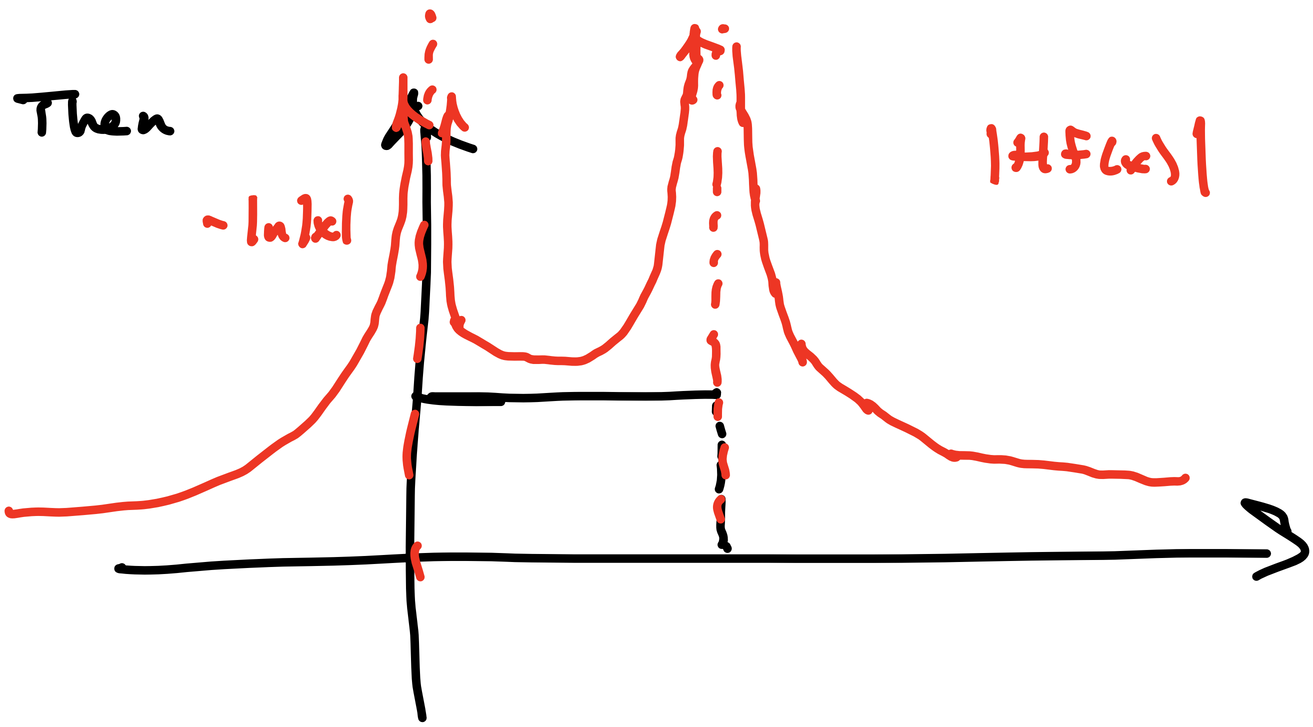
- If $f \in L^1$, $Hf \in$ weak-L^1.
- If $f \in L^p$, $Hf \in L^p$.

Q: What about $f \in L^\infty$?

Example:

Consider $f = \chi_{[0,1]}(x)$

Then



Functions with logarithmic singularities are L^p for all $p \in [1, \infty)$. So $|HF(x)|$ belongs to a space between L^∞ and L^p for $p < \infty$. This space is called the space of functions having bounded Mean oscillation.

<u>Def:</u>

Let $f \in L^1_{loc}(\mathbb{R})$. Define

$$f^\#(x) := \sup_{I \ni x} \frac{1}{|I|} \int |f(y) - f_I| \, dy$$

where

$$f_I := \frac{1}{|I|} \int_I f$$

$f \in BMO(\mathbb{R})$ if and only if $f^\# \in L^\infty(\mathbb{R})$

and

$$\|f\|_{BMO(\mathbb{R})} := \|f^\#\|_{L^\infty(\mathbb{R})}.$$

Properties of BMO space

- $\| f(\lambda \cdot) \|_{BMO} = \|f\|_{BMO}$

- Let $I_2 \subset I_1$, where I_2 is a child of I_1.

  Then

  $$|f_{I_2} - f_{I_1}|$$

  $$= \left| \frac{1}{|I_2|} \int_{I_2} f - f_{I_1} \right| \leq \frac{|I_1|}{|I_2|} \int_{I_2} |f - f_{I_1}|$$

  $$\leq 2 \|f\|_{BMO}.$$

Let $I_1 \supset I_2 \supset \cdots \supset I_n$ be a sequence of direct descendents then

$$|f_{I_n} - f_{I_1}| \leq n \, 2 \|f\|_{BMO}.$$

- $L^\infty \subsetneq BMO$

Let g be the "discrete logarithm"

$$g(x) = \sum_{n=0}^{\infty} \chi_{[0,2^{-n}]}(x).$$

Then $g \notin L^\infty(\mathbb{R})$, but

for I centered at zero with $|I|<1$,

$$\frac{1}{|I|}\int_I g = \frac{1}{|I|}\int \sum \chi_{[0,2^{-n}]} = \frac{1}{|I|}\sum_{2^{-n}<|I|} 2^{-n} + \sum_{2^{-n}>|I|} \frac{1}{|I|}\frac{1}{2}|I|$$

$$\sim \log(|I|^{-1}).$$

$$\Rightarrow g^{\#}(0) = \sup_{I \ni 0} \frac{1}{|I|}\int_I |g(x) - g_I|\,dx$$

$$\sup_{I \ni 0} \frac{1}{|I|}\int_I |g(x) - \log(|I|^{-1})|\,dx.$$

$$\sup_{I \ni 0} \frac{1}{|I|}\int_I \left| \sum_{n=1}^{\infty} \chi_{[0,2^{-n}]} - \log(|I|^{-1}) \right| dx.$$

on I, $\sum \chi_{[0,2^{-n}]} \gtrsim \log(|I|^{-1})$

thus

$$g^{\#}(0) \lesssim \sup_{I \ni 0} \left(\frac{1}{|I|} \int_I \sum_{n=1}^{\infty} \chi_{[0,2^{-n}]} - \log(|I|^{-1}) \, dx \right)$$

$$\lesssim \sup_{I \ni 0} \log(c|I|^{-1}) - \log(|I|^{-1})$$

$$\sim \sup_{I \ni 0} \log(c)$$

Claim $\chi_{\{|x|<1\}} \log|x| \in BMO([-1,1])$ and

$\chi_{\{|x|<1\}} \operatorname{sign}(x) \log|x| \notin BMO([-1,1])$

- Let $f^{*}(x) := \sup_{I \ni x} \inf_{c \in \mathbb{R}} \frac{1}{|I|} \int |f(y) - c| \, dy.$

Then if $f \in L^1_{loc}(\mathbb{R})$

$$f^{\#}(x) \sim f^{*}(x)$$

Obviously, $f^{*}(x) \leq f^{\#}(x)$

Let $c \in \mathbb{R}$, then

$$f^{\#}(x) = \sup_{I \ni x} \frac{1}{|I|} \int |f(y) - f_I| \, dy$$

Thus,

$$f^{\#}(x) \leq \sup_{I \ni x} \frac{1}{|I|} \int |f(y) - c| \, dy + \sup_{I \ni x} \frac{1}{|I|} \int |f_I - c| \, dy.$$

$$= \sup_{I \ni x} \frac{1}{|I|} \int |f(y) - c| \, dy + \sup_{I \ni x} |f_I - c|$$

$$\leq 2 \sup_{I \ni x} \int |f(y) - c| \, dy.$$

Since c is arbitrary,

$$f^{\#}(x) \leq 2 f^{*}(x)$$

Therefore,

$$\|f\|'_{BMO} \sim \|f^{*}\|_{L^\infty}.$$

Thm:

For $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (or $f \in C_c(\mathbb{R})$)

$$\|Hf\|_{BMO} \lesssim \|f\|_\infty.$$

Pf: Consider $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$

and $B(x_0, r) \subset \mathbb{R}$

define

$$c_0 := \int_{|x_0-y|>2r} \frac{1}{x_0-y} f(y)\, dy$$

Then

$$\int_{B(x_0,r)} |Hf(x) - c_0|\, dx \leq \int_{B(x_0,r)} \int_{|x_0-y|>2r} \left| \frac{1}{x-y} - \frac{1}{x_0-y} \right| |f(y)|\, dy\, dx$$

$$+ \int_{B(x_0,r)} H(\chi_{B(x_0,2r)} f)$$

$$\lesssim |B(x_0,r)| \|f\|_\infty$$

$$+ |B(x_0,r)|^{1/2} \|H(\chi_{B(x_0,2r)} f)\|_{L^2}$$

$$\lesssim \|f\|_\infty \qquad \square$$