A Generalized Family of Operators

It is perhaps not obvious that we've used a few specific properties of the kernel of the Hilbert transform. This allows for a straightforward generalization to what are called Calderón-Zygmund kernels.

<u>Def:</u> Let $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$, let $B > 0$.

If K satisfies

i.) $|K(x)| \leq B|x|^{-d}$, for all $x \in \mathbb{R}^d \setminus \{0\}$

ii.) $\displaystyle\int_{\{|x| > 2|y|\}} |K(x) - K(x-y)|\,dx \leq B \quad \forall y \in \mathbb{R}^d \setminus \{0\}$

iii.) $\displaystyle\int_{r<|x|<s} K(x)\,dx = 0$ for all $0 < r < s < \infty$.

Then K is called a <u>Calderón-Zygmund kernel</u>.

The operator, $Tf := K * f$, $f \in \mathcal{S}(\mathbb{R}^d)$ is called a <u>Calderón-Zygmund operator</u>

<u>Thm:</u> Let T be a Calderón-Zygmund operator. Then for $1 < p < \infty$

$$\|T\|_{p \to p} < \infty.$$

and T is $L^1 \to$ weak-L^1 bounded.

<u>pf.</u> Exercise. (Paint-by-numbers)

Before we start to build the theory for pointwise convergence of partial sum operators, let's discuss pointwise convergence for the Hilbert transform.

We note that although we can show that $Hf \in L^p(\mathbb{R})$, when $f \in L^p(\mathbb{R})$, when $f \in L^p \setminus S$, Hf is defined only abstractly by continuous extension. Specifically, we can't necessarily say that for $f \in L^p \setminus S$

$$Hf(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{1}{x-y} f(y)\, dy.$$

This is similar to extending the classical derivative operator to weak derivatives.

To this end, we define a maximal operator for the Hilbert transform. Denote

$$H_* f(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y|>\varepsilon} \frac{1}{x-y} f(y)\,dy \right|.$$

<u>Prop:</u>

H_* satisfies

$$H_* f(x) \leq M(Hf)(x) + Mf(x)$$

<u>Pf:</u>

Let $\tilde{K}(x) = \chi_{\{|x| \geq 1\}} K(x)$

and for $\varepsilon > 0$

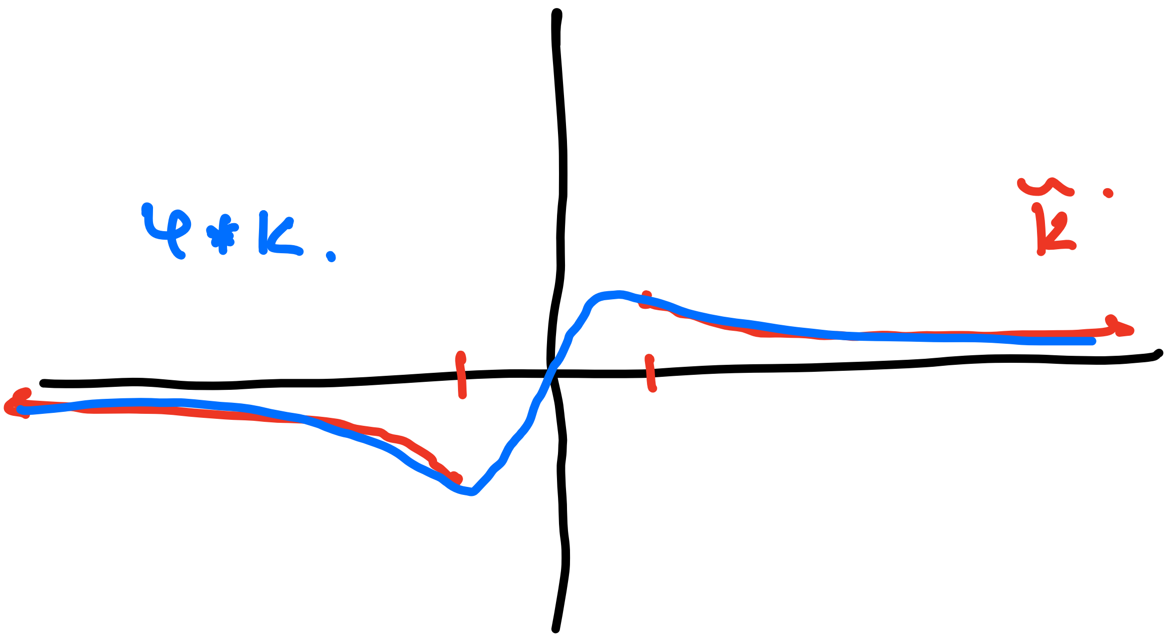
$$\tilde{K}_\varepsilon(x) = \varepsilon^{-d} \tilde{K}\left(\tfrac{x}{\varepsilon}\right) = \chi_{\{|x| \geq \varepsilon\}} K(x).$$

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ satisfy $\varphi \geq 0$ and

$\int \varphi = 1$.

Let

$$\Phi := \varphi * K - \hat{K}.$$



Then

$$\Phi_\varepsilon = (\varphi * K)_\varepsilon - \hat{K}_\varepsilon = \varphi_\varepsilon * K_\varepsilon - \hat{K}_\varepsilon$$
$$= \varphi_\varepsilon * K - \hat{K}_\varepsilon.$$

and

$$\hat{K}_\varepsilon * f = \varphi_\varepsilon * (K * f) - \Phi_\varepsilon * f.$$

Claim: $\{\Phi_\varepsilon\}$ and $\{\Psi_\varepsilon\}$ are radially bounded approximate identities.

Pf of claim: $\Phi(x) = \int \varphi(y)\left(\frac{1}{x-y}\right)dy - \int \varphi(y)\frac{1}{x}\,dy$

$$= \int \varphi(y)\frac{y}{x(x-y)}\,dy.$$

$$\lesssim \frac{1}{x^2}\int \varphi(y)\,dy.$$

Therefore, we've previously proved that

$$(H_* f)(x) = \sup_{\varepsilon>0}\left|(\hat{K}_\varepsilon * f)(x)\right|$$

$$\lesssim \left(\sup_\varepsilon \|\varphi_\varepsilon\|_1\right) M(K*f) + \left(\sup_\varepsilon \|\Phi_\varepsilon\|_1\right) Mf$$

$$\lesssim M(Hf) + Mf$$

$\square$.

Corollary: H_* is bounded on $L^p(\mathbb{R})$ for $1<p<\infty$. Also, H_* is weak-L^1 bounded.