Thm: For every $f \in S(\mathbb{R})$, H is weak-L^1 bounded. I.e.

$$|\{x \in \mathbb{R} \mid |Hf(x)| > \lambda\}| \lesssim \frac{1}{\lambda} \|f\|_1 \quad \forall \lambda > 0.$$

Pf:

Let $\lambda > 0$,

we perform a Calderón-Zygmund decomposition of f at height λ.

Then $f = g + b$, $|g| \leq \lambda$ and

$$b = \sum_{I \in \mathcal{B}} \chi_I f \quad , \quad \sum_{I \in \mathcal{B}} |I| \leq \frac{1}{\lambda} \|f\|_1$$

Now we modify g and b. Define

$$f_1 := g + \sum_{I \in \mathcal{B}} \chi_I \frac{1}{|I|} \int_I f$$

$$f_2 := b - \sum_{I \in \mathcal{B}} \chi_I \frac{1}{|I|} \int_I f.$$

Then

$$f_2 = \sum_{I \in \mathcal{B}} \chi_I \cdot \left(f - \frac{1}{|I|}\int_I f\right).$$

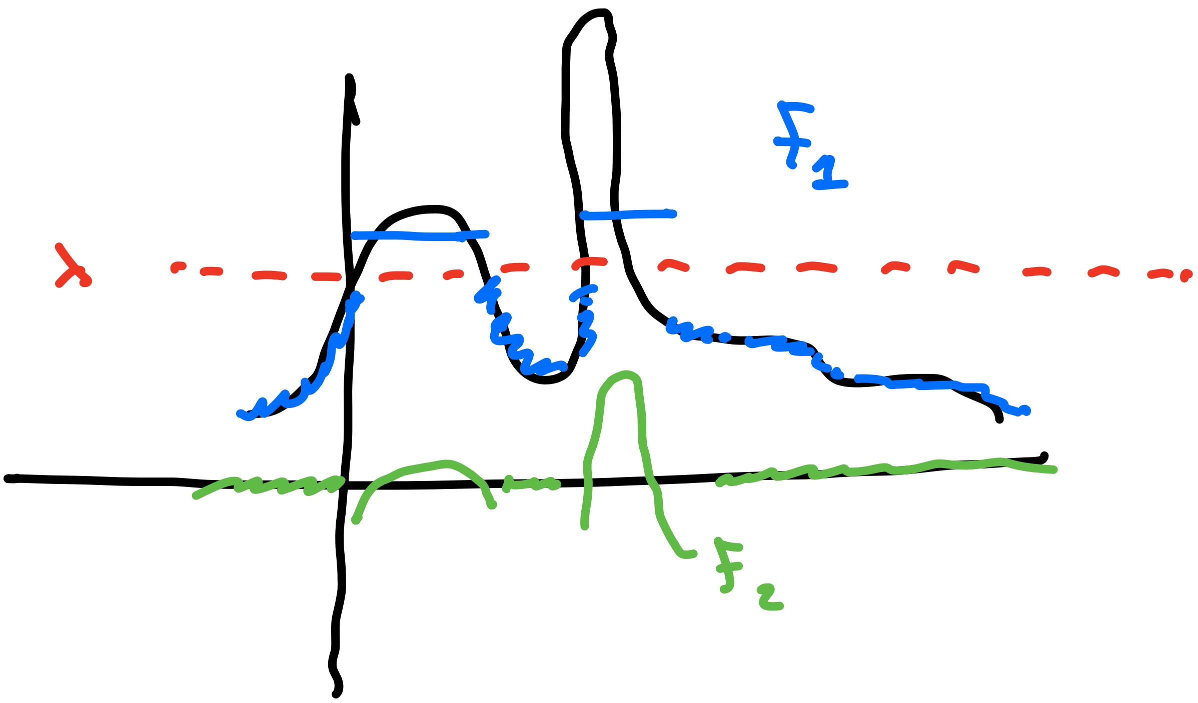
$$=: \sum_{I \in \mathcal{B}} f_I$$

where $$f_I := \chi_I \left(f - \frac{1}{|I|}\int_I f\right).$$

$\boxed{\text{Note: } \int f_I = 0.}$

Also, $\|f_1\|_\infty \le 2\lambda$, $\|f_1\|_1 \le \|f\|_1$

and $\|f_2\|_2 \le 2\|f\|_2$.



Now, since H is L^2-bounded,

$$\left|\{x\in\mathbb{R} \mid |Hf(x)|>\lambda\}\right|$$

$$\leq \left|\{x\in\mathbb{R} \mid |Hf_1(x)|>\tfrac{\lambda}{2}\}\right| + \left|\{x\in\mathbb{R} \mid |Hf_2(x)|>\tfrac{\lambda}{2}\}\right|$$

$$\lesssim \frac{\|Hf_1\|_2^2}{\lambda^2} + \left|\{|Hf_2|>\tfrac{\lambda}{2}\}\right|$$

$$\lesssim \frac{\|f_1\|_2^2}{\lambda^2} + \left|\{|Hf_2|>\tfrac{\lambda}{2}\}\right|$$

$$\leq \frac{\|f_1\|_\infty \|f_1\|_1}{\lambda^2} + \left|\{|Hf_2|>\tfrac{\lambda}{2}\}\right|$$

$$\lesssim \frac{1}{\lambda}\|f\|_1 + \left|\{|Hf_2|>\tfrac{\lambda}{2}\}\right|$$

Now for the second term, we first define the dilated intervals

$I^* = 2\cdot I.$

Then

$$\left|\left\{|Hf_2| > \tfrac{\lambda}{2}\right\}\right|$$

$$\leq \left|\bigcup_{I\in\mathcal{B}} I^*\right| + \left|\left\{x\in\mathbb{R}\setminus\bigcup_{I\in\mathcal{B}} I^* : |Hf_2(x)| > \tfrac{\lambda}{2}\right\}\right|$$

$$\lesssim \sum_{I\in\mathcal{B}} |I| + \frac{1}{\lambda}\int_{\mathbb{R}\setminus\bigcup I^*} |Hf_2|$$

$$\lesssim \frac{1}{\lambda}\|f\|_1 + \frac{1}{\lambda}\sum_{I\in\mathcal{B}} \int_{\mathbb{R}\setminus I^*} |Hf_I| .$$

Let y_I = midpoint of the interval, I.

Since $\int f_I = 0$,

$$Hf_I(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} \frac{1}{x-y} f_I(y)\,dy$$

$$= \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} \left(\frac{1}{x-y} - \frac{1}{x-y_I}\right) f_I(y)\,dy$$

$$= \lim_{\varepsilon\to 0}\int_{\substack{|x-y|>\varepsilon\\ y\in I}} \left(\frac{y-y_I}{(x-y)(x-y_I)}\right) f_I(y)\,dy$$

For $x \in \mathbb{R} \setminus I^*$,

$$\lim_{\varepsilon \to 0} \int_{\substack{|x-y|>\varepsilon \\ y \in I}} \left(\frac{y - y_I}{(x-y)(x-y_I)} \right) f_I(y)\, dy$$

$$= \int_I \frac{y - y_I}{(x-y)(x-y_I)} f_I(y)\, dy .$$

Thus,

$$\int_{\mathbb{R} \setminus I^*} |Hf_I(x)|\, dx \le \int_{\mathbb{R} \setminus I^*} \int_I \frac{|y-y_I|}{|x-y||x-y_I|} |f_I(y)|\, dy\, dx$$

$$= \int_I \int_{\mathbb{R} \setminus I^*} \frac{|y-y_I|}{|x-y||x-y_I|} |f_I(y)|\, dx\, dy$$

$|y - y_I| \le |I|$ and

$$\int_{\mathbb{R} \setminus I^*} \frac{1}{|x-y|\,|x-y_I|} \lesssim \frac{1}{|I|}$$

$\Rightarrow$

$$\int_{\mathbb{R} \setminus I^*} |Hf_I(x)| \lesssim \int_I |f_I(y)|\, dy$$

$$\lesssim 2 \int_I |f(y)|\, dy .$$

Therefore,

$$\left|\{ |Hf_2| > \lambda/2 \}\right|$$

$$\lesssim \frac{1}{\lambda}\|f\|_1 + \sum_{I \in \mathcal{B}} \frac{1}{\lambda} \int_I |f|$$

$$\lesssim \frac{1}{\lambda}\|f\|_{L^1}$$

Putting everything together, we get

$$\left|\{|Hf| > \lambda\}\right| \leq \left|\{|Hf_1| > \lambda/2\}\right| + \left|\{|Hf_2| > \lambda/2\}\right|$$

$$\lesssim \frac{\|f\|_1}{\lambda} \qquad \square.$$

<u>Corollary</u>:

For $1 < p < \infty$, $\|H\|_{p \to p} < \infty$.

<u>Pf</u>: H is weak-L^1 bounded and L^2-bounded. By Marcinkiewicz interpolation, for $p \in (1, 2]$,

$$\|H\|_{p \to p} < \infty.$$

Now, observe that for $f, g \in S(\mathbb{R})$

$$\langle Hf, g\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} Hf \cdot \overline{g}$$

$$= \int_{\mathbb{R}} (K*f)(x)\, \bar{g}(x)\, dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y)\, K(x-y)\, dy\ \bar{g}(x)\, dx$$

Since $K(z) = -K(-z)$

$$= \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} K(x-y)\, \bar{g}(x)\, dx\ dy.$$

$$= -\int_{\mathbb{R}} f(y) \int_{\mathbb{R}} K(y-x)\, \bar{g}(x)\, dx\ dy$$

$$= -\langle f, K*g\rangle_{L^2(\mathbb{R})}$$

$$= -\langle f, Hg\rangle_{L^2(\mathbb{R})}.$$

For $2 < p < \infty$.

$$\|Hf\|_{L^p} = \sup_{\|g\|_{p'}=1} |\langle Hf, g\rangle|$$

where $\frac{1}{p} + \frac{1}{p'} = 1$

$$= \sup_{\|g\|_{p'}=1} |\langle f, Hg\rangle|$$

$$\leq \|f\|_{L^p}\ \|H\|_{p' \to p'}$$

$\square$.