L^2-boundedness of the Hilbert Transform

We will estimate the Fourier-transform of $\chi_{\{|x|>\varepsilon\}}(x)\frac{1}{x}$, then use Plancherel to prove L^2-boundedness.

Lemma:

Let $K_\varepsilon(x) = \chi_{\{|x|>\varepsilon\}}(x)\cdot\frac{1}{x}$. Then

$$\sup_{\varepsilon>0}\sup_{\xi\neq 0}\left|\widehat{K_\varepsilon}(\xi)\right| \lesssim 1.$$

pf: Suppose, $\xi \neq 0$

$$\widehat{K_\varepsilon}(\xi) = \lim_{R\to\infty}\int_{|x|\leq R} e(-x\xi)\,K_\varepsilon(x)\,dx.$$

$$= \int_{|x|\leq|\xi|^{-1}} e(-x\xi)\,K_\varepsilon(x)\,dx$$

$$+ \lim_{R\to\infty}\int_{|\xi|^{-1}\leq|x|\leq R} e(-x\xi)\,K_\varepsilon(x)\,dx.$$

WLOG, assume $\varepsilon < |\xi|^{-1}$, then

$$\int_{|x|\le|\xi|^{-1}} e(-x\xi)\, K_\varepsilon(x)\, dx$$

$$= \int_{\varepsilon<|x|\le|\xi|^{-1}} e(-x\xi)\,\frac{1}{x}\, dx$$

$$= \int_{\varepsilon<|x|\le|\xi|^{-1}} e(-x\xi)\,\frac{1}{x}\,dx - \int_{\varepsilon<|x|\le|\xi|^{-1}} \frac{1}{x}\,dx.$$

$$= \int_{\varepsilon<|x|\le|\xi|^{-1}} [e(-x\xi)-1]\,\frac{1}{x}\,dx.$$

Observation:

For $|x\xi|\le 1$, $|e(-x\xi)-1| \sim |x\xi|$.

$\Rightarrow$

$$\left|\int_{\varepsilon<|x|\le|\xi|^{-1}} [e(-x\xi)-1]\frac{1}{x}\,dx\right| \le \int_{\varepsilon<|x|\le|\xi|^{-1}} |\xi|\,dx$$

$$\lesssim 1.$$

For the second term, note that

$e^{\pi i} = -1$, and

$$\int_{|\xi|^{-1} < |x| \leq R} e^{2\pi i x \xi} \frac{1}{x} dx = \int_{|\xi|^{-1} < |x - \frac{1}{2\xi}| \leq R} e^{2\pi i x \xi} e^{\pi i} \frac{1}{x - \frac{1}{2\xi}} dx$$

$$= -1 \int_{|\xi|^{-1} < |x - \frac{1}{2\xi}| \leq R.} e(x\xi) \frac{1}{x - \frac{1}{2\xi}} dx$$

Therefore,

$$\int_{|\xi|^{-1} < |x| \leq R} e^{2\pi i x \xi} \frac{1}{x} dx$$

$$= \frac{1}{2} \left(\int_{|\xi|^{-1} < |x| \leq R} e^{2\pi i x \xi} \frac{1}{x} dx - \int_{|\xi|^{-1} < |x - \frac{1}{2\xi}| \leq R} e^{2\pi i x \xi} \frac{1}{x - \frac{1}{2\xi}} dx \right)$$

$$= \frac{1}{2} \int_{|\xi|^{-1} < |x| \leq R} e^{2\pi i x \xi} \left(\frac{1}{x} - \frac{1}{x - \frac{1}{2\xi}} \right) - \frac{1}{2} \int_{\{|\xi|^{-1} < |x - \frac{1}{2\xi}| \leq R\} \Delta \{|\xi|^{-1} < |x| \leq R\}} e^{2\pi i x \xi} \frac{1}{x - 1/2\xi} dx$$

$$= \frac{1}{2}\int_{|\xi|^{-1}<|x|\le R} e^{2\pi i x\xi}\left(\frac{-1/2\xi}{x(x-1/2\xi)}\right) - \frac{1}{2}\int_{\{|\xi|^{-1}<|x-\frac{1}{2\xi}|<R\}\,\Delta\,\{|\xi|^{-1}<|x|\le R\}} e^{2\pi i x\xi}\frac{1}{x-\frac{1}{2\xi}}\,dx$$

Therefore,

$$\left|\int_{|\xi|^{-1}<|x|\le R} e^{2\pi i x\xi}\frac{1}{x}\,dx\right|$$

$$\lesssim \left(|\xi|^{-1}\right)^{-1}\cdot\frac{1}{|\xi|} + \frac{1}{|\xi|}\frac{1}{R} + |\xi|^{-1}\cdot|\xi| + \frac{1}{R}|\xi|^{-1}$$

$$\Rightarrow \sup_{\xi}\lim_{R\to\infty}\left|\int_{|\xi|^{-1}<|x|\le R} e^{2\pi i x\xi}\frac{1}{x}dx\right| \lesssim 1.$$

$\square$.

<u>Corollary</u>

$$\|H\|_{2\to 2}<\infty$$

<u>pf:</u> By Plancherel, let $f\in C_c^\infty(\mathbb{R})$.

$$\|Hf\|_2 = \lim_{\varepsilon\to 0}\|H_\varepsilon f\|_2 = \lim_{\varepsilon\to 0}\|K_\varepsilon * f\|_2 \le \limsup_{\varepsilon\to 0}\|\hat{K}_\varepsilon\hat{f}\|_2.$$

$$\le \left(\limsup_{\varepsilon\to 0}\|\hat{K}_\varepsilon\|_\infty\right)\|f\|_2$$

$\square$.

Weak-L^1 boundedness of the Hilbert Transform

We are now prepared to prove weak-L^1 boundedness. We will actually use L^2-boundedness to achieve this.

The decay of the kernel of the Hilbert transform should remind us of the maximal function operator. But the maximal function operator is not locally singular like the Hilbert transform. So the argument will look familiar, but will be very different.

The Calderón-Zygmund Decomposition

Lemma:

Let $f \in L^1(\mathbb{R})$ and $\lambda > 0$. Then one can write $f = g + b$ with $|g| \leq \lambda$ and $b = \sum_{I \in \mathcal{B}} \chi_I f$ where $\mathcal{B} = \{I\}$ is a collection of disjoint intervals such that, for each I one has

$$\lambda \leq \frac{1}{|I|} \int_I |f| \leq 2\lambda.$$

Furthermore,

$$\Big| \bigcup_{I \in \mathcal{B}} I \Big| < \frac{1}{\lambda} \cdot \|f\|_{L^1}.$$

Pf: For each $k \in \mathbb{Z}$, define the collection of dyadic intervals, $\mathcal{D}_k$, by

$$\mathcal{D}_k := \{ [2^k m, 2^k(m+1)) \mid m \in \mathbb{Z} \}.$$

Observation: Let $I_1 \in \mathcal{D}_{k_1}$ and $I_2 \in \mathcal{D}_{k_2}$ then one of the following must hold

- $I_1 \cap I_2 = \emptyset$
- $I_1 \subset I_2$
- $I_2 \subset I_1$.

There exists k_0 large enough so that

$$\frac{1}{|I|}\int_I |f| \le \lambda \quad \text{for all } I \in \mathcal{D}_{k_0}$$

For each $I \in \mathcal{D}_{k_0}$, there are two intervals J_1 and J_2 s.t. $J_1, J_2 \in \mathcal{D}_{k_0+1}$ and $J_1, J_2 \subset I$. We call these the children of I. Then either

$$\frac{1}{|J_i|}\int_{J_i} |f| \le \lambda \quad \text{or} \quad \frac{1}{|J_i|}\int_{J_i} |f| > \lambda$$

for each i.

- If $\frac{1}{|J_i|}\int_{J_i} |f| > \lambda$, then let $J_i \in \mathcal{B}$ and note that

$$\frac{1}{|J_i|}\int_{J_i} |f| \le \frac{1}{|J_i|}\int_I |f| = \frac{2}{|I|}\int_I |f| \le 2\lambda.$$

- If $\frac{1}{|J_i|}\int_{J_i}|f| \le \lambda$, then we subdivide J_i and continue.

Now
$$\Big|\bigcup_{I\in\mathcal{B}} I\Big| \le \sum_{I\in\mathcal{B}}|I| < \sum_{\mathcal{B}}\frac{1}{\lambda}\int_I|f| \le \frac{1}{\lambda}\|f\|_1.$$

and if $x \in \mathbb{R}\setminus\bigcup_{I\in\mathcal{B}} I$

then $\exists\{I_m\}_{m=1}^{\infty}$ s.t. $I_m \to x$

and $\frac{1}{|I_m|}\int_{I_m}|f| \le \lambda$

Thus, by Lebesgue differentiation,

$$|f(x)| \le \lambda \quad \text{for a.e. } x \in \mathbb{R}\setminus\bigcup_{I\in\mathcal{B}} I.$$

Let $g := f - \sum_{I\in\mathcal{B}}\chi_I f$

then $|g| \le \lambda$ a.e. $\square$.