Thm: Let $\{\Phi_T\}_{T>0}$ be a radially bounded Approximate Identity.

Then for any $f \in L^1$

$$\Phi_T * f \xrightarrow{T\to\infty} f \quad \text{a.e}.$$

Pf: Let $\varepsilon > 0$. Suppose $g \in C_c(\mathbb{R})$

$\|f-g\|_1 < \varepsilon^2$. Recall that $\|\Phi_T * g - g\|_\infty \xrightarrow{T\to\infty} 0$.

Thus,

$$\left|\left\{x\in\mathbb{R} \;\middle|\; \limsup_{T\to\infty} |\Phi_T * f(x) - f(x)| > \varepsilon\right\}\right|$$

$$\leq \left|\left\{\limsup_{T\to\infty} |\Phi_T * f - \Phi_T * g| > \tfrac{1}{3}\varepsilon\right\}\right|$$

$$+ \underbrace{\left|\left\{\limsup_{T\to\infty} |\Phi_T * g - g| > \tfrac{1}{3}\varepsilon\right\}\right|}_{=0}$$

$$+ \left|\left\{\limsup_{T\to\infty} |f-g| > \tfrac{1}{3}\varepsilon\right\}\right|$$

$$\lesssim \left|\left\{M(f-g) > \tfrac{1}{3}\varepsilon\right\}\right| + \left|\left\{|f-g| > \tfrac{1}{3}\varepsilon\right\}\right|$$

$$\lesssim \frac{\|f-g\|_1}{\varepsilon} < \frac{\varepsilon^2}{\varepsilon} < \varepsilon.$$

$\blacksquare$

The Hilbert Transform.

For any $f \in \mathcal{S}(\mathbb{R})$, $\varepsilon > 0$, consider the operator

$$H_\varepsilon f(x) := \int_{|x-y|>\varepsilon} \frac{1}{x-y} f(y)\, dy.$$

$$= \int_{|y|>\varepsilon} \frac{1}{y} f(x-y)\, dy.$$

The Hilbert Transform is defined by taking the limit of H_ε as $\varepsilon \to 0$.

$$\boxed{Hf(x) := \lim_{\varepsilon \to 0} H_\varepsilon f(x).}$$

The first objective is showing that Hf exists for $f \in \mathcal{S}(\mathbb{R})$.

Observe that for some $R > 0$

$$\int_{|y|>\varepsilon} \frac{1}{y} f(x-y)\, dy = \int_{R>|y|>\varepsilon} \frac{1}{y} f(x-y)\, dy + \varepsilon.$$

Note that

$$\int_{R>|y|>\varepsilon} \frac{1}{y}\, dy = 0 \quad ,$$

so we can resort to an old trick:

$$\int_{2^{-k}>|y|>2^{-(k+1)}} \frac{1}{y} f(x-y)\, dy = \int_{2^{-k}>|y|>2^{-(k+1)}} \left(\frac{1}{y} f(x-y) - \frac{1}{y} f(x)\right) dy$$

$$= \int_{2^{-k}>|y|>2^{-(k+1)}} \frac{1}{y}\left(f(x-y) - f(x)\right) dy$$

$$\lesssim \int_{2^{-k}>|y|>2^{-(k+1)}} \frac{1}{|y|} \|f'\|_\infty |y|$$

$$\lesssim 2^{-k} \|f'\|_\infty$$

for all $k \geq 1$.

Thus

$Hf(x)$ exists for all $x \in \mathbb{R}$ for $f \in C_c^\infty(\mathbb{R})$, and $\|Hf\|_\infty \lesssim \|f'\|_\infty$.

Now we can justify the Hilbert Transform's utility for the partial sum convergence question.

Let $K(x) = \frac{1}{x}$, then recall that

$$D_T(x) = \frac{\sin(2\pi x T)}{\pi x} = \frac{e(xT) - e(-xT)}{2i\pi x}$$

which is very similar to K.

In fact,

$$D_T(x) = \chi_{[-\frac{1}{T}, 1/T]}(x) D_T(x) + \chi_{[-\frac{1}{T}, \frac{1}{T}]^c}(x) D_T(x)$$

$$=: D_T^1(x) + D_T^2(x)$$

and

$$D_T^2(x) = \chi_{[-\frac{1}{T}, \frac{1}{T}]^c}(x) \frac{e(xT)}{2i\pi x} - \chi_{[-\frac{1}{T}, \frac{1}{T}]^c}(x) \frac{e(-xT)}{2i\pi x}$$

Then

$$(D_T^2 * f)(x) = \int_{|y| > \frac{1}{T}} \frac{e(yT)}{2i\pi y} f(x-y)\,dy$$

$$- \int_{|y|>\frac{1}{T}} \frac{e(-yT)}{2i\pi y} f(x-y)\,dy$$

$$= e(xT) \frac{1}{2i\pi} \int_{|y|>\frac{1}{T}} \frac{1}{y} \left[e(-T(x-y)) f(x-y) \right] dy$$

$$- e(xT) \cdot \frac{1}{2i\pi} \int_{|y|>\frac{1}{T}} \frac{1}{y} \left[e(T(x-y)) f(x-y) \right] dy$$

If we let

$$E_T f(x) := e(xT) f(x) \quad , \text{then}$$

$$(D_T^2 * f)(x) = -\frac{i}{2\pi} E_T \left[H_{\frac{1}{T}} (E_{-T} f - E_T f) \right].$$

We have shown that

$$\sup_{T>0} \|S_T f\|_p \le \sup_{T>0} \|D^1_T * f\|_p + \sup_{T>0} \|D^2_T * f\|_p$$

$$\le \left(\sup_{T>0} \|D^1_T\|_1\right) \|f\|_p$$

$$+ \left(\sup_{T>0} \|E_T\|_{p\to p}\right)\left(\sup_{T>0} \|H_{1/T}\|_{p\to p}\right)$$

$$\cdot \left(\sup_{T>0} \|E_{-T}\|_{p\to p} + \sup_{T>0} \|E_T\|_{p\to p}\right)$$

$$\cdot \|f\|_p.$$

Now it is clear that

$$\sup_{T>0} \|D^1_T\|_1 < \infty \quad \text{and}$$

$$\sup_{T>0} \|E_T\|_{p\to p}, \quad \sup_{T>0} \|E_{-T}\|_{p\to p} < \infty.$$

Therefore, it suffices to show that

$$\sup_{T>0} \|H_{1/T}\|_{p\to p} = \sup_{\varepsilon>0} \|H_\varepsilon\|_{p\to p} < \infty.$$