Singular Integral Theory

In order to prove L^p-convergence, we will need to use the theory of Calderon and Zygmund. We start with the Hardy-Littlewood Maximal Function as a toy example.

Def: (Hardy-Littlewood Maximal Function) Let $f \in L^1_{loc}(\mathbb{R})$. For any $x \in \mathbb{R}$, define

$$Mf(x) := \sup_{x \in I \subset \mathbb{R}} \frac{1}{|I|} \int_I |f(x)|\, dx.$$

- $M(f+g) \leq Mf + Mg$
- $M(cf) = |c|\, M(f)$, $c \in \mathbb{R}$.

Covering Lemmas

Lemma: Let $\{I_k\}_{k=1}^N$ be a finite collection of intervals. There exists a subcollection of intervals, $\{J_\ell\}_{\ell=1}^M \subset \{I_k\}$, such that

- $J_{\ell_1} \cap J_{\ell_2} = \phi$ for all $\ell_1 \neq \ell_2$
- $\bigcup_{k=1}^N I_k \subset \bigcup^M 3 \cdot J_\ell$.

Pf:

Consider a maximal interval, J_1,

By induction, assume that

$$\{J_\ell\}_{\ell=1}^m \text{ is a disjoint subcollection s.t.}$$

$$\bigcup_{I_k \cap J_\ell \neq \phi} I_k \subset \bigcup_{\ell=1}^m 3 \cdot J_\ell$$

Let J_{m+1} = maximal interval in

$$\{I_k \mid I_k \cap J_\ell = \phi \text{ for all } \ell = 1, \dots, m\}$$

Since $\{I_k\}$ is a finite collection, this process will end.

By construction,

$$\bigcup_{k=1}^{N} I_k \subset \bigcup_{\ell=1}^{M} 3 \cdot J_\ell$$

$\square$.

Corollary:

Let $E \subset \mathbb{R}$ and let $\{I_\alpha\}_\alpha$ be an open cover of E. Then for any $c < |E|$, there exists a disjoint, finite subcollection of intervals, $\{J_\ell\}$, s.t.

$$\sum |J_\ell| \geq \frac{c}{3}.$$

Prop: The Hardy-Littlewood maximal operator, M, satisfies:

- M is bounded for $L^1(\mathbb{R})$ to weak-L^1, i.e.

$$|\{x \in \mathbb{R} \mid Mf(x) > \lambda\}| \leq \frac{3}{\lambda} \|f\|_1.$$

- For $p \in (1, \infty]$

$$\|M\|_{L^p \to L^p} < \infty.$$

Pf: Weak-L^1 bound

Fix $\lambda > 0$ and a compact, $K \subset \{x \in \mathbb{R} \mid Mf(x) > \lambda\}$.

To each $x \in K$, $\exists I_x$ s.t. I_x is an open interval, $x \in I_x$, $\int_{I_x} |f(x)| dx > \lambda |I_x|$.

Since K is compact, there is a finite subcover, $\{I_k\}_{k=1}^N$. Moreover, we have the following claim:

Claim: Let $\{I_k\}$ be a finite collection of intervals. There exist a subcollection, $\{J_\ell\}$, s.t. $J_{\ell_1} \cap J_{\ell_2} = \emptyset$ for $\ell_1 \neq \ell_2$ and

$$\bigcup I_k \subset \bigcup 3 \cdot J_\ell.$$

(In higher dimensions, we replace 3 with 5 and call this the 5r-covering lemma)

Thus

$$K \subset \bigcup 3 \cdot J_\ell$$

$$\Rightarrow |K| \leq \left|\bigcup 3J_\ell\right| \leq 3 \sum_\ell |J_\ell|$$

$$\leq \frac{3}{\lambda} \sum \int_{J_\ell} |f| \leq \frac{3\|f\|_1}{\lambda}.$$

ii.) We observe that

$$\|Mf\|_\infty \leq \|f\|_\infty.$$

Now by Marcinkiewicz interpolation

$$\|M\|_{p \to p} < \infty \quad \text{for } 1 < p \leq \infty.$$

Almost Everywhere convergence of Approximate Identities

We are still on our way to proving L^p convergence of Fourier Series.

First, we prove a pointwise convergence theorem that demonstrates the utility of the Maximal Function.

Def: (Radially Bounded Approximate Identity). Let $\{\Phi_T\}_{T>0}$ be an approximate identity. We say that it is radially bounded if there exists $\{\Psi_T\}_{T>0}$ s.t.

- $|\Phi_T| \leq \Psi_T$.
- Ψ_T is even and decreasing
- $\sup_{T>0} \|\Psi_T\|_1 < \infty$.

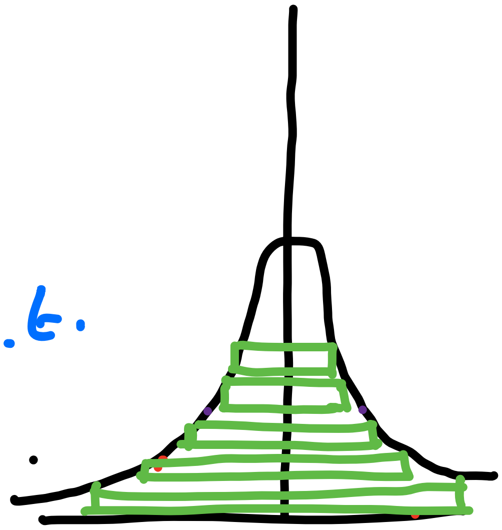
Lemma: If $\{\Phi_T\}_{T>0}$ is a radially bounded approximate identity, then for any $f \in L^1_{loc}(\mathbb{R})$

$$\sup_{T>0} \left|(\Phi_T * f)(x)\right| \le \left(\sup_{T>0} \|\psi_T\|_1\right) \cdot Mf(x).$$

Pf.

First, observe that

$$\left|(\Phi_T * f)(x)\right| \le (\psi_T * |f|)(x)$$



Claim: $\exists\, \mu_T \in \mathcal{M}([0,\infty))$ s.t.

$$\psi_T(x) = \int_0^\infty \chi_{[-t,t]}(x)\, \mu(dt)$$

and $\displaystyle\int \psi_T(x)\,dx = \int_0^1 2t\, \mu(dt)$

Then

$$\psi_T * |f| = \int_0^\infty \chi_{[-t,t]} * f \quad \mu(dt)$$

$$= \int_0^\infty \frac{1}{2t}\, \chi_{[-t,t]} * f \quad 2t\, \mu(dt)$$

$$\le Mf \cdot \int_0^\infty 2t\, \mu(dt). \qquad \square$$