Properties of the Dirichlet Kernel

$$D_T(x) = \int_{-T}^{T} e(x\xi)\, d\xi$$

$$= \frac{1}{2\pi i x}\left[e(xT) - e(-xT)\right]$$

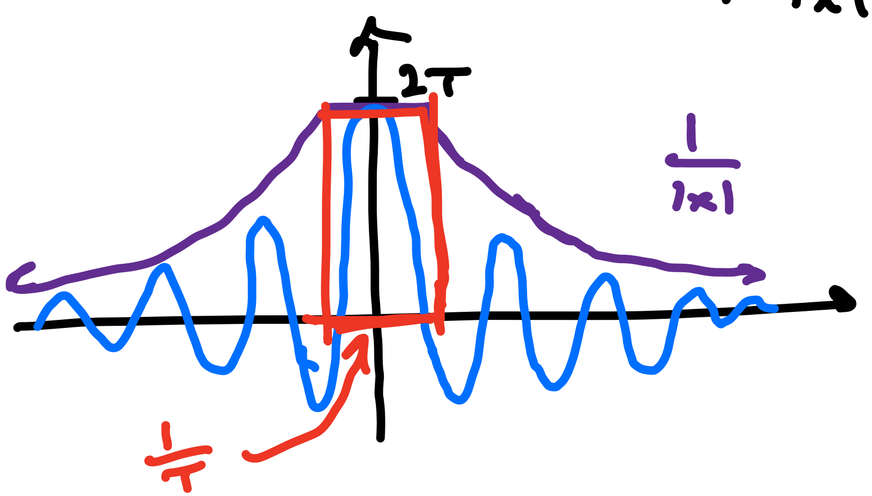
$$= \frac{1}{2\pi i x}\, 2i \sin(2\pi x T)$$

$$= \frac{1}{\pi x} \sin(2\pi x T).$$

First, $|D_T(x)| \le \int_{-T}^{T} 1\, d\xi \le 2T.$

and $|D_T(x)| \le \dfrac{|\sin(2\pi x T)|}{|\pi x|} \le \dfrac{1}{\pi} \dfrac{1}{|x|}.$

$\Rightarrow \quad |D_T(x)| \le C \min\left(T, \frac{1}{|x|}\right).$



Near the origin, D_T resembles the approximate identity $T\chi_{[-\frac{1}{T},\frac{1}{T}]}$, but the tails of D_T decay critically slowly.

Therefore, $\|D_T\|_{L^1} \sim \log(T)$ for $T>1$.

This suggests that

$\|D_T * f - f\|_{L^p}$ does not necessarily converge to zero for $f \in L^p(\mathbb{R})$, $p\in[1,\infty]$.

This would indeed be the case if

$$D_T(x) = \min(T, |x|^{-1})$$

However, the actual form of $D_T(x)$ has enough cancellation to make the reality complicated.

We notice that the question of whether

$$\|D_T * f - f\|_{L^p} \to 0 \quad \text{for } f \in L^p$$

is not the same question as whether

$$D_T * f \longrightarrow f \quad \text{Lebesgue a.e. for } f \in L^p.$$

The first question is a lot easier to answer, so we deal with that first.

Let's start with a comparison to kernels smoother than the Dirichlet kernel.

Formally,

$$\widehat{D_T}(\xi) = \chi_{[-T,T]}(\xi).$$

The relationship between the discontinuity of $\chi_{[-T,T]}$ and the relatively slow decay of $|D_T(x)|$.

If we average the partial sums operator or the Dirichlet kernel we get a smoother kernel.

For $f \in L^1(\mathbb{R})$, $T > 0$, let

$$\sigma_T f(x) := \frac{1}{T}\int_0^T S_t f(x)\, dt.$$

If

$$K_T := \frac{1}{T}\int_0^T D_t \, dt \quad , \text{ then}$$

$$\sigma_T f = K_T * f \quad \text{and}$$

K_T is known as the Fejér kernel

$$K_T(x) = \frac{1}{T}\int_0^T \int_{-t}^{t} e(sx)\,ds\,dt.$$

$$= \frac{1}{T}\int_0^T \frac{1}{\pi x} \sin(2\pi t x)\,dt.$$

$$= \frac{1}{T} \frac{-1}{2\pi^2 x^2}\left(\cos(2\pi T x) - 1\right)$$

$$= \frac{1}{T} \frac{-1}{2\pi^2 x^2} \quad -2\sin^2(\pi T x)$$

$$= \frac{1}{T} \frac{\sin^2(\pi T x)}{\pi^2 x^2}.$$

$$0 \le K_T(x) \lesssim \frac{1}{T}\min\left(T^2, \frac{1}{x^2}\right)$$

Moreover, $K_T(x) = \frac{1}{T}\left(D_T(x)\right)^2$.

<u>Exercise</u>: $\{K_T(x)\}_{T>0}$ is an approximate identity.

Lemma (Riemann-Lebesgue Lemma)

If $f \in L^1(\mathbb{R})$, the $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

pf: Given $\varepsilon > 0$, $\exists T > 0$ s.t.
$\| \sigma_T f - f \|_1 < \varepsilon$. For ξ satisfying
$|\xi| > T$, $|\hat{f}(\xi)| = |\widehat{\sigma_T f}(\xi) - \hat{f}(\xi)|$
$\leq \| \sigma_T f - f \|_1 < \varepsilon$. $\square$.

$\alpha \in (0,1)$

Thm: Let $f \in C_c^\alpha(\mathbb{R})$.

Then $\| S_T f - f \|_\infty \to 0$ as $T \to \infty$

pf. First, note that $\int K_T = 1$

thus

$$\sigma_T f(x) - f(x) = (K_T * f)(x) - f(x) \int K_T(y)\,dy.$$

$$= \int f(x-y) K_T(y)\,dy - \int f(x) K_T(y)\,dy$$

$$= \int (f(x-y) - f(x))\, K_T(y)\, dy.$$

Recall that $|K_T(y)| \le \frac{1}{T} \min(T^2, \frac{1}{y^2})$

$$\le \frac{1}{T} \frac{1}{y^2}.$$

and $|f(x-y) - f(x)| \lesssim [f]_\alpha |y|^\alpha$

thus

$$|\sigma_T f(x) - f(x)| \lesssim \frac{1}{T}\left(\int_{(\operatorname{supp} f + x) \setminus \{|x| < \frac{1}{T}\}} \frac{|y|^\alpha}{y^2} + T^2 \int_{|x| < \frac{1}{T}} \frac{1}{T^\alpha} \right) \lesssim T^{-\alpha}$$

$\Rightarrow$

$$\|\sigma_T f - f\|_\infty \lesssim T^{-\alpha}$$

Observe that

$$S_T f - f = S_T(f - \sigma_T f) + \sigma_T f - f$$

$\Rightarrow$

$$\|S_T f - f\|_\infty \lesssim \left(\|D_T\|_{L^2([-c,c])} + 1 \right) \|f - \sigma_T f\|$$

$$\lesssim_f T^{-\alpha} \log(T) \to 0$$

$\square$.

L^p Convergence ..

Prop: The following statements are equivalent for any $1 \le p < \infty$.

(i) For every $f \in L^p(\mathbb{R})$

$$\| S_T f - f \|_p \to 0 \quad T \to \infty.$$

(ii) $\sup \| S_T \|_{p \to p} < \infty$

(Also, $\| S_T f - f \|_{L^\infty_{loc}} \to 0 \ \forall f \in C_c(\mathbb{R}) \iff \sup \| S_T \|_{L^\infty_{loc} \to L^\infty_{loc}} < \infty$.

Pf:

(ii) ⇒ (i)

Suppose $\sup_T \| S_T \|_{p \to p} < \infty$.

Then recall that $\sigma_T f = S_T \sigma_T f$.

Therefore,

$$S_T f - f = S_T f - S_T \sigma_T f + \sigma_T f - f.$$

$$\Rightarrow \| S_T f - f \|_p \le \left(\sup_T \| S_T \| + 1 \right) \| \sigma_T f - f \|_p \xrightarrow{T \to \infty} 0.$$

(i) ⇒ (ii). Uniform Boundedness Principle.

$\square$

Cor: The Fourier Series do not converge on $C([-1,1])$ and $L^1([-1,1])$

pf: It suffices to show that

$$\sup_T \|S_T\|_{L^\infty(I)\to L^\infty(I)} = \infty$$

$$\sup_T \|S_T\|_{L^1(I)\to L^1(I)} = \infty.$$

Then

$$\|S_T\|_{L^\infty \to L^\infty} \geq \sup_{g\in C([-1,1])} \|S_T g\|_\infty$$

$$\geq \sup_{g\in C([-1,1])} |(D_T * g)(0)|$$

$$= \|D_T\|_{L^1([-1,1])}. \to \infty.$$

Similarly, if $\{\Phi_\delta\}$ is an approx. identity.

$$\|S_T\|_{L^1\to L^1} = \sup_{\delta>0} \|D_T * \Phi_\delta\|_{L^1(I)} = \|D_T\|_{L^1(I)} \xrightarrow{T\to\infty} \infty.$$

$\square$