For fixed $I \in \Delta_n$

$$\sum_{J \in \Delta} |\langle h_I, T_0 h_J \rangle| \lesssim \sum_{k=1}^{2^n} \frac{1}{k^{1+\delta}} \leq C(\delta)$$

Thus, by Schur's test

$$\|\Delta_n T_0 \Delta_n\|_{2\to 2} \leq C(\delta) \quad \text{for all } n.$$

Next, Recall Cotlar's Lemma

Lemma: (Cotlar's Lemma)
Let $\{T_j\}_{j=1}^{N}$, $T_j : \mathcal{H} \to \mathcal{H}$, $\gamma : \mathbb{Z} \to \mathbb{R}^+$ s.t.

$$\|T_j^* T_k\| \leq \gamma^2(j-k),$$

$$\|T_j T_k^*\| \leq \gamma^2(j-k).$$

If $\sum_{k \in \mathbb{Z}} \gamma(k) \leq A < \infty$, then

$$\left\|\sum_{j=1}^{N} T_j\right\|_{\mathcal{H}\to\mathcal{H}} \leq A.$$

For each $n \in \mathbb{Z}_+$,

$\mathbb{E}_n T_0 \Delta_n$ is a transformation from the set of functions spanned by $\{h_I\}_{I \in \mathcal{D}_n}$ into the set of functions spanned by

$$\{|J|^{-1/2} \chi_J\}_{J \in \mathcal{D}_n}$$

Thus $E_n T_0 \Delta_n$ has a matrix representation with terms

$$\langle T_0 h_I , |J|^{-1/2} \chi_J \rangle$$

$$= |J|^{-1/2} \left[\langle T_0 h_I , \chi_J \chi_{2\cdot I} \rangle + \langle T_0 h_I , \chi_J \chi_{(2\cdot I)^c} \rangle \right]$$

$$\leq |J|^{-1/2} \|T_0 h_I\|_2 \min(|J|^{1/2}, |I|^{1/2}) \, \chi_{\{2\cdot I \cap J \neq \emptyset\}}(J)$$

$$+ C \int_{J \setminus 2\cdot I} \frac{|I|^{1/2+\delta} |J|^{-1/2}}{\operatorname{dist}(x, I)^{1+\delta}} \, dx .$$

$$\lesssim \chi_{\{2\cdot I \cap J \neq \emptyset\}}(J) \min(|J|^{+1/2}, |I|^{1/2}) \cdot |J|^{-1/2}$$

$$+ \frac{|I|^{1/2+\delta}}{\operatorname{dist}(J, I)^{1+\delta}} |J|^{-1/2} \cdot |J| \, \chi_{\{2\cdot I \cap J = \emptyset\}}(J)$$

Then

$$\sup_{I \in \mathcal{D}_n} \sum_{J \in \mathcal{D}_n} \chi_{\{2\cdot I \cap J \neq \emptyset\}}(J)$$

$$+ \frac{|I|^{1/2+\delta}}{\operatorname{dist}(J, I)^{1+\delta}} |J|^{1/2} \cdot \chi_{\{2\cdot I \cap J = \emptyset\}}(J)$$

$$\leq \sup_{I \in \mathcal{D}_n} \left(3 + \sum_{k=1}^{2^n} \frac{|I|^{1/2+\delta} (2^{-n})^{1/2}}{(|I| + k\, 2^{-n})^{1+\delta}.} \right)$$

$$\leq \sup_{I \in \mathcal{D}_n} \quad 3 + \sum_{m=1}^{n} |I|^{1/2+\delta} (2^{n})^{1/2+\delta} C(\delta)$$

$$\lesssim C(\delta) \quad .$$

and

$$\sup_{J \in \mathcal{D}_{\leq n}} \sum_{I \in \mathcal{D}_n} \chi_{\{2 \cdot I \cap J \neq \phi\}}(I) + \frac{|I|^{1/2+\delta} |J|^{1/2}}{\operatorname{dist}(J, I)^{1+\delta}} \cdot \chi_{\{2 \cdot I \cap J = \phi\}}(I)$$

$$\lesssim \sup_{J \in \mathcal{D}_{\leq n}} \left(3 + \sum_{k=1}^{2^n} |I|^{1/2+\delta-(1+\delta)} |J|^{1/2} \frac{1}{(1+k)^{1+\delta}.} \right)$$

$$\lesssim C(\delta)$$

Thus, by Schur's Test

$$\sup_n \| E_n T_0 \Delta_n \|_{2 \to 2} \leq C(\delta).$$

(The same argument implies
$\sup_n \|\Delta_n T_0 \mathbb{E}_n\|_{2\to 2} = \sup_n \|\mathbb{E}_n T_0^* \Delta_n\|_{2\to 2} \le C(\delta)$)

In order to apply Cotlar's lemma we now need to estimate

$$\|\mathbb{E}_n T_0 \Delta_n (\mathbb{E}_m T_0 \Delta_m)^*\|_{2\to 2} \quad \text{and}$$

$$\|(\mathbb{E}_m T_0 \Delta_m)^* \mathbb{E}_n T_0 \Delta_n\|_{2\to 2}. \quad \text{for } n \neq m.$$

By symmetry, it suffices to let $m < n$ and estimate

$$\|(\mathbb{E}_n T_0 \Delta_n)^* \mathbb{E}_m T_0 \Delta_m\|_{2\to 2}.$$

Let

$$\begin{aligned} S_{nm} &:= (\mathbb{E}_n T_0 \Delta_n)^* \mathbb{E}_m T_0 \Delta_m = \Delta_n T_0^* \mathbb{E}_n \mathbb{E}_m T_0 \Delta_m \\ &= \Delta_n T_0^* \mathbb{E}_{n \wedge m} T_0 \Delta_m = \Delta_n T_0^* \mathbb{E}_m T_0 \Delta_m. \end{aligned}$$

where $n \wedge m = \min(n, m) = m$

S_{nm} is a transformation from the set of functions spanned by $\{h_I\}_{I \in \mathcal{D}_n}$

into the space of functions spanned by $\{h_J\}_{J \in \mathcal{D}_m}$ with coefficients

$$\langle \Delta_n T_0^* \mathbb{E}_m T_0 \Delta_m h_I, h_J \rangle$$

$$= \langle \mathbb{E}_n T_0 h_I, \mathbb{E}_m T_0 h_J \rangle$$

We first use the simplest estimate.

$$|\mathbb{E}_n T_0 h_I(y)| \lesssim |I|^{-1/2} \frac{|I|^{1+\delta}}{(|I| + \mathrm{dist}(y, I))^{1+\delta}}.$$

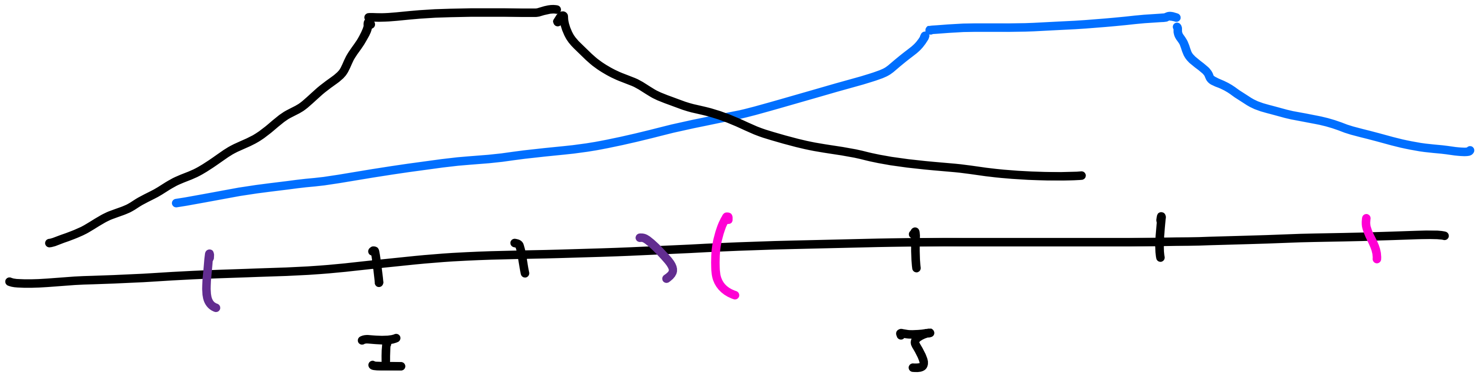
$$|\mathbb{E}_m T_0 h_J(y)| \lesssim |J|^{-1/2} \frac{|J|^{1+\delta}}{(|J| + \mathrm{dist}(y, J))^{1+\delta}}$$

This is due to decay away the support of I and J along with the estimate

$$\fint_I T_0 h_I \leq |I|^{-1/2} \|T_0 h_I\|_2 \leq |I|^{-1/2}.$$

Therefore,

$$|\langle \mathbb{E}_n T_0 h_I, \mathbb{E}_m T_0 h_J \rangle| \lesssim \int_{\mathbb{R}} (|I|\,|J|)^{-1/2} \frac{|I|^{1+\delta}\,|J|^{1+\delta}}{(|I| + \mathrm{dist}(y, I))^{1+\delta} \cdot (|J| + \mathrm{dist}(y, J))^{1+\delta}}$$



Let $$\varphi_J(y) = \frac{|J|^{1+\delta}}{(|J| + dist(y, J))^{1+\delta}}$$

$$\varphi_I(y) = \frac{|I|^{1+\delta}}{(|J| + dist(y, I))^{1+\delta}}.$$

Note that $\|\varphi_J\|_{L^1} \leq |J|$ and $\|\varphi_I\|_{L^1} \leq |I|$

and $\varphi_J \sim \frac{1}{|J|} \chi_J * \varphi_{J_0}$ and

$\varphi_I \sim \frac{1}{|I|} \chi_I * \varphi_{I_0}$

Thus,

$$\sup_{I\in\mathcal{D}_n} \sum_{J\in\mathcal{D}_m} |\langle E_n T_0 h_I , E_m T_0 h_J \rangle|$$

$$\leq \sup_{I\in\mathcal{D}_n} \sum_{J\in\mathcal{D}_m} (|I||J|)^{-1/2} \langle \varphi_I , \varphi_J \rangle$$

$$\sim \sup_{I\in\mathcal{D}_n} \sum_{J\in\mathcal{D}_m} \frac{(|I||J|)^{-1/2}}{|I||J|} \langle \varphi_{I_0} * \chi_I , \varphi_{J_0} * \chi_J \rangle$$

$$= \sup_{I\in\mathcal{D}_n} \sum_{J\in\mathcal{D}_m} \frac{(|I||J|)^{-1/2}}{|I||J|} \langle \chi_I , \varphi_{I_0} * \varphi_{J_0} * \chi_J \rangle .$$

$$= \sup_{I\in\mathcal{D}_n} \frac{(|I| 2^{-m})^{-1/2}}{|I| 2^{-m}} \langle \chi_I , \varphi_{I_0} * \varphi_{J_0} * \sum_J \chi_J \rangle .$$

Since $\| \varphi_{I_0} * \varphi_{J_0} * \sum_J \chi_J \|_\infty \leq \| \varphi_{I_0} * \varphi_{J_0} \|_{L^2} \leq \| \varphi_{I_0} \|_{L^2} \| \varphi_{J_0} \|_{L^2}$

$$\leq |I| \, 2^{-m}.$$

Moreover,

$$\sup_{I\in\mathcal{D}_n} \sum_{J\in\mathcal{D}_m} |\langle E_n T_0 h_I , E_m T_0 h_J \rangle|$$

$$\lesssim \sup_{I\in\mathcal{D}_n} \frac{(|I| \, 2^{-m})^{-1/2}}{|I| \, 2^{-m}} \, |I| \; |I| \, 2^{-m}$$

$$\lesssim (2^{m-n})^{1/2}$$

The next estimate is much harder, but we have the $T_0(1)$ condition.

$$\sup_{J\in\mathcal{D}_m}\sum_{I\in\mathcal{D}_n}|\langle \mathbb{E}_n T_0 h_I, \mathbb{E}_m T_0 h_J\rangle|$$

$$= \sup_{J\in\mathcal{D}_m}\sum_{\substack{I\in\mathcal{D}_n\\ \mathrm{dist}(I,\partial J)\le 2^{(n-m)(1-\delta)}|I|}}|\langle \mathbb{E}_m T_0 h_I, T_0 h_J\rangle + \sum_{\substack{I\in\mathcal{D}_n\\ \mathrm{dist}(I,\partial J)\ge 2^{(n-m)(1-\delta)}|I|}}|\langle \mathbb{E}_m T_0 h_I, T_0 h_J\rangle|$$

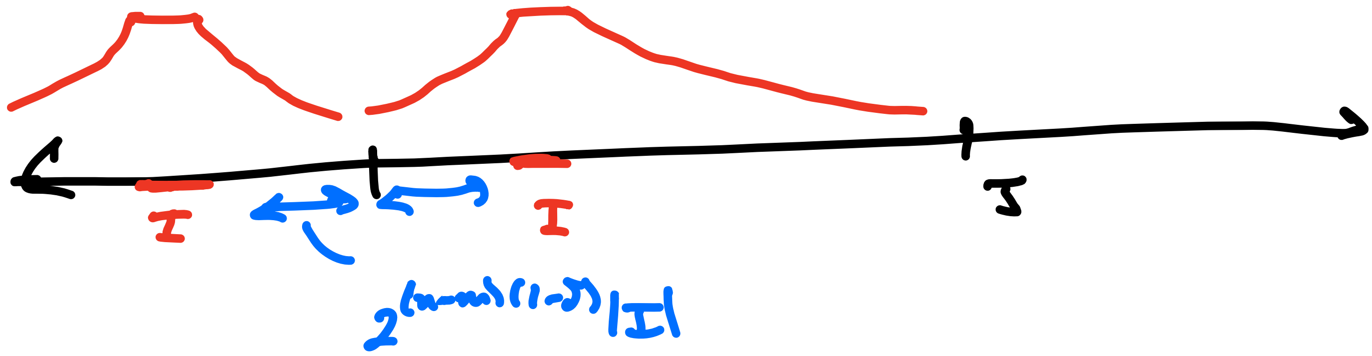
Then, as before

$$\sup_{J\in\mathcal{D}_m}\sum_{I\in\mathcal{D}_n}|\langle \mathbb{E}_n T_0 h_I, \mathbb{E}_m T_0 h_J\rangle|$$

$$\le \sup_{J\in\mathcal{D}_m}\frac{(2^{-n}|J|)^{-1/2}}{2^{-m}|J|}\left\langle \sum_{\substack{I\in\mathcal{D}_n\\ \mathrm{dist}(I,\partial \tilde{I})\ge 2^{(n-m)(1-\delta)}}} \chi_I, \varphi_{I_0}*\varphi_{J_0}*\chi_J\right\rangle$$

$$\lesssim \sup_{J\in\mathcal{D}_m}(2^{-m}|J|)^{-1/2}|I|\,2^{(n-m)(1-\delta)}$$

$$= \left(2^{n-m}\right)^{1/2-\delta}$$



For $I \in \mathcal{D}_n$ and $I \subset J$ since $\langle 1, T_0 h_I \rangle = 0$

$$\int_J T_0 h_I = \langle \chi_J, T_0 h_I \rangle$$

$$= \langle \chi_{J^c}, T_0 h_I \rangle$$

$$\le \int_{J^c} |I|^{-1/2} \psi_I$$

For $\mathrm{dist}(I, \partial J) \ge 2^{(n-m)(1-\delta)} |I|$

$$\le |I|^{-1/2} |I|^{1+\delta} \left(2^{(n-m)(1-\delta)} |I|\right)^{-\delta}$$

Now

$$\sum_{\substack{I \subset J \\ \mathrm{dist}(I,\partial J) > 2^{(n-m)(1-\delta)}|I|}} |\langle E_m T_0 h_I, E_n T_0 h_J \rangle|$$

$$\le \sum_{\substack{I \subset J \\ \mathrm{dist}(I,\partial J) > 2^{(n-m)\delta}|I|}} \left| \int_J T_0 h_I \fint_J T_0 h_J \right| + \int_{J^c} |T_0 h_I| \, |T_0 h_J|$$

$$\le \sum_{\substack{I \subset J \\ \mathrm{dist}(I,\partial J) > 2^{(n-m)(1-\delta)}|I|}} \int_{J^c} |T_0 h_I| \fint_J |T_0 h_J| + \int_{J^c} (|I||J|)^{-1/2} |T_0 h_I| \, \|\psi_J\|_{\infty}$$

$$\leq \sum_{\substack{I \subset J \\ \operatorname{dist}(I, \partial J) > 2^{(n-m)(1-\delta)}|I|}} |J|^{-1/2} |I|^{1/2} 2^{-(n-m)(1-\delta)\delta} + \int_{J^c} (|I||J|)^{-1/2} \varphi_I$$

$$\leq 2^{-(n-m)1/2} 2^{(n-m)(1-(1-\delta)\delta)} + 2^{-(n-m)1/2} 2^{(n-m)(1-(1-\delta)\delta)}$$

$$\lesssim 2^{(n-m)(1-(1-\delta)\delta - 1/2)}$$

$$= 2^{(n-m)(1/2 - (1-\delta)\delta)}$$

And

$$\sum_{\substack{I \in \mathcal{D}_n \\ \operatorname{dist}(I, J) \geq 2^{(n-m)(1-\delta)}|I|}} |\langle \mathbb{E}_n T_0 h_I, \mathbb{E}_m T_0 h_J \rangle|$$

$$\lesssim \sum_{\substack{I \in \mathcal{D}_n \\ \operatorname{dist}(I, J) \geq 2^{(n-m)(1-\delta)}|I|}} (|I||J|)^{-1/2} \langle \varphi_{I_0} * \chi_I, \varphi_J \rangle$$

$$\leq (|I_0||J_0|)^{-1/2} \int_{\operatorname{dist}(x, J) \geq 2^{(n-m)(1-\delta)}|I_0|} \varphi_{I_0} * \varphi_J .$$

$$\leq (|I_0||J_0|)^{-1/2} \max \left(|I| \frac{|J_0|^{1+\delta}}{(2^{(n-m)(1-\delta)}|I_0|)^{1+\delta}}, \frac{|J_0||I|^{1+\delta}}{(2^{(n-m)(1-\delta)}|I_0|)^{1+\delta}} \right)$$

$$= |I_0|^{1/2} |J|^{-1/2} \left[2^{(n-m)(1-\delta)} \, 2^{-(n-m)} \right]^{1+\delta}$$

$$= \left(2^{(n-m)}\right)^{-1/2} \left(2^{n-m}\right)^{\delta(1+\delta)}$$

$$= 2^{(n-m)\left(-1/2 + \delta(1+\delta)\right)}$$

Finally,

$$\| (\mathbb{E}_m T_0 \Delta_m)^* \Delta_n T_0 \mathbb{E}_n \|_{L^2 \to L^2}$$

$$\leq \left(\sup_{I \in \mathcal{D}_n} \sum_{J \in \mathcal{D}_m} |\langle \mathbb{E}_n T_0 h_I, \mathbb{E}_m T_0 h_J \rangle| \right)^{1/2}$$

$$\cdot \left(\sup_{J \in \mathcal{D}_m} \sum_{I \in \mathcal{D}_n} |\langle \mathbb{E}_n T_0 h_I, \mathbb{E}_m T_0 h_J \rangle| \right)^{1/2}.$$

$$\lesssim \left[2^{(m-n)1/2} \right]^{1/2} \left[2^{(n-m)(\frac{1}{2}-\delta)} \right]^{1/2}$$

$$\lesssim \quad 2^{-(n-m)\delta/2} .$$

This estimate is strong enough to employ Cotlar's Lemma. Thus

$$\left\|\sum \mathbb{E}_n T \Delta_n\right\|_{2\to 2}, \left\|\sum \Delta_n T \mathbb{E}_n\right\|_{2\to 2} < \infty \quad \square$$