We are now ready to prove $T(1)$

Thm: (T(1) Theorem).

Let T be a singular integral operator such that $T, T^*: \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R}) \cap L^1_{loc}(\mathbb{R})$.

Assume that

(1) $\max\left(|\langle h_I, T^*\varphi\rangle|, |\langle h_I, T\varphi\rangle|\right) \leq \|\varphi\|_2$

for all $\varphi \in \mathcal{S}(\mathbb{R})$, $I \in \mathcal{D}$.

(2) $\exists C > 0$ s.t.

$$|\langle \chi_{[0,1]}, T^*a\rangle| + |\langle \chi_{[0,1]}, Ta\rangle| \leq C$$

for all smooth $\mathcal{H}^1([0,1])$ atoms a.

The T is bounded on $L^2([0,1])$

Pf:

Let $T(\chi_{[0,1]}) =: b_1$ and

$T^*(\chi_{[0,1]}) =: b_2$. (By assumption $b_1, b_2 \in BMO$)

Define the operator

$$T_0 := T - \pi_{b_1} - \pi^*_{b_2}$$

Then $T_0(\chi_{[0,1]}) = T_0^*(\chi_{[0,1]}) = 0$.

We need decay estimates for this new operator. Let $I \in \mathcal{D}$ and $y \in [0,1] \setminus 2 \cdot I$.

Since $\pi_{b_1} h_I(y) = \pi^*_{b_2} h_I(y) = 0$ for $y \notin I$.

Therefore,

$T_0 h_I(y) = T h_I(y)$ and $T_0^* h_I(y) = T h_I(y)$

for $y \in [0,1] \setminus 2 \cdot I$.

Thus,

$$|T_0 h_I(y)| = |T h_I(y)| = \left| \int K(x,y)\, h_I(x) \right|$$

$$= \left| \int \big(K(x,y) - K(x_I, y)\big)\, h_I(x) \right|$$

$$\leq |I|^{+1/2} \sup_{x \in I} \sup_{y \in [0,1] \setminus 2 \cdot I} |K(x,y) - K(x_I, y)|$$

$$\leq |I|^{+1/2} \cdot \sup_{x \in I} \sup_{y \in [0,1] \setminus 2 \cdot I} \frac{|x - x_I|^{\delta}}{|x-y|^{1+\delta}}$$

$$\leq C \frac{|I|^{+1/2}\, |I|^{\delta}}{[|I| + \mathrm{dist}(y,I)]^{1+\delta}} \quad \text{for } y \in [0,1] \setminus 2I$$

and similarly

$$|T_0^* h_I(y)| \leq C \frac{|I|^{+1/2+\delta}}{[|I| + \mathrm{dist}(y,I)]^{1+\delta}}$$

Recall that $\Delta_n = \mathbb{E}_{n+1} - \mathbb{E}_n$, thus for any $f \in L^2$

$$f = E_0(f) + \sum_{n=0}^{\infty} \Delta_n f.$$

$$\text{and} \quad \Delta_n f = \sum_{I \in \mathcal{D}_n} \langle f, h_I \rangle h_I$$

Note that since $T_0 1 = 0 = T_0^* 1$.

$$T_0 E_0 f = T_0(\xi f) = 0$$
$$\text{and } T_0^* E_0 f = T_0^*(\xi f) = 0$$
for any $f \in L^2$.

And for any $g \in L^2$, $f \in L^2$.

$$\langle E_0(T_0 g), f \rangle = \langle T_0 g, E_0(f) \rangle = \langle g, T_0^* E_0(f) \rangle$$
$$= 0$$

$\Rightarrow \quad E_0(T_0 g) = 0$ for any $g \in L^2$.

Thus,

$$T_0 f = T_0(E_0 f + \Sigma \Delta_n f) = T_0(\Sigma \Delta_n f)$$

$$= E_0 T(\Sigma \Delta_n f) + \sum_m \Delta_m (T_0(\Sigma \Delta_n f))$$

$$= \sum_m \Delta_m \left[T_0 \left(\sum_n \Delta_n f \right) \right]$$

$$= \sum_{n,m} \Delta_m T_0 \Delta_n f$$

$$= \sum_n \Delta_n T_0 \Delta_n f + \sum_{m > n} \Delta_m T_0 \Delta_n f + \sum_{n > m} \Delta_m T_0 \Delta_n f$$

$$= \sum_{n=0}^{\infty} \Delta_n T_0 \Delta_n f + \sum_{m=0}^{\infty} \Delta_m T_0 E_m f + \sum_{n=0}^{\infty} E_n (T_0 \Delta_n f)$$

We handle each term separately, applying Schur's Test and Cotlar's Lemma.

First,

$$(\Delta_n T_0 \Delta_n)^* = \Delta_n T_0^* \Delta_n$$

$\Rightarrow$ for $n \neq m$,

$$(\Delta_n T_0 \Delta_n)^* (\Delta_m T_0 \Delta_m) = 0 = (\Delta_m T_0 \Delta_m)(\Delta_n T_0 \Delta_n)^*.$$

Thus,

$$\Big\| \sum_{n=0}^{\infty} \Delta_n T_0 \Delta_n \Big\|_{2\to 2} \leq \sup_{n \geq 0} \| \Delta_n T_0 \Delta_n \|_{2 \to 2}$$

Fix n.

Then $\Delta_n T_0 \Delta_n$ is an operator on the finite-dimensional space spanned by $\{h_I\}_{I \in \mathcal{D}_n}$.

Now for $I, J \in \mathcal{D}_n$

$$|\langle h_I, T_0 h_J \rangle| \leq \langle w_I, |T_0 h_J| \rangle \, |I|^{-1/2}.$$

$$\leq |I|^{1/2} \sup_{y \in I \cap ([0,1] \setminus 2J)} |T_0 h_J(y)|$$

$$\leq |I|^{1/2} \frac{|J|^{1/2+\delta}}{\operatorname{dist}(I,J)^{1+\delta}}$$

For fixed $I \in \Delta_n$

$$\sum_{J \in \Delta} |\langle h_I, T_0 h_J \rangle| \lesssim \sum_{k=1}^{2^n} \frac{1}{k^{1+\delta}} \leq C(\delta)$$

Thus, by Schur's test

$$\|\Delta_n T_0 \Delta_n\|_{2\to 2} \leq C(\delta) \quad \text{for all } n.$$

Next, Recall Cotlar's Lemma

Lemma: (Cotlar's Lemma)

Let $\{T_j\}_{j=1}^{N}$, $T_j : \mathcal{H} \to \mathcal{H}$, $\gamma : \mathbb{Z} \to \mathbb{R}^+$ s.t.

$$\|T_j^* T_k\| \leq \gamma^2(j-k),$$

$$\|T_j T_k^*\| \leq \gamma^2(j-k).$$

If $\sum_{k \in \mathbb{Z}} \gamma(k) \leq A < \infty$, then

$$\left\|\sum_{j=1}^{N} T_j\right\|_{\mathcal{H}\to\mathcal{H}} \leq A.$$

For each $n \in \mathbb{Z}_+$,

$\mathbb{E}_n T_0 \Delta_n$ is a transformation from the set