Paraproducts

Def: Given $b \in BMO([0,1])$ and $f \in L^2$, we define the paraproduct between b and f as

$$\pi_b(f) = \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \fint_I f \cdot h_I$$

Note: $\pi_b(1) = b$, $\pi_b^*(1) = 0$.

Exercise: for $x \notin I$, $\pi_b(h_I) = \pi^*_b(h_I) = 0$.

Prop: π_b is bounded on $L^2([0,1])$. In particular,

$$\|\pi_b(f)\|_{L^2} \lesssim \|b\|_{BMO} \|f\|_{L^2}.$$

Furthermore, π_b is bounded on $L^p([0,1])$ for $p \in (1,\infty)$.

pf: Suppose $\|b\|_{BMO} = 1$.

We first observe that

$$\|\pi_b(f)\|_{L^2}^2 = \sum_{I \in \mathcal{D}} |\langle b, h_I \rangle|^2 \left| \fint_I f \right|^2.$$

By the definition of BMO,

for every $J \in \mathcal{D}$

$$|J|^{-1} \sum_{L \subset J} |\langle b, h_L \rangle|^2 \le \|b\|_{BMO} = 1.$$

$$\Rightarrow \quad \sum_{L \subset J} |\langle b, h_L \rangle|^2 \le |J|.$$

Thus, the function defined on $\mathcal{D}$ by

$$I \longmapsto |\langle b, h_I \rangle|^2$$

satisfies the Carleson condition.
The Carleson condition proposition now implies

$$\sum_{I \in \mathcal{D}} |\langle b, h_I \rangle|^2 \left| \fint_I f \right|^2$$

$$\lesssim \int_0^1 \sup_{I \ni x} \left| \fint_I f \right|^2$$

$$\le \| Mf \|_2^2 \lesssim \|f\|_2^2 .$$

Thus, we have shown

$$\|\pi_b(f)\|_{L^2} \lesssim \|b\|_{BMO} \|f\|_{L^2}$$

Now, we need to show weak-L^1 boundedness. Since $supp(\pi_b(f)) \subset supp(f)$, we can perform of C-Z decomp of an L^1 function, f, at height $\lambda > 0$, to obtain $f = f_1 + f_2$

Then since π_b is a linear operator

$$\left|\{|\pi_b(f)| > \lambda\}\right| \le \left|\{|\pi_b(f_1)| > \tfrac{\lambda}{2}\}\right| + \left|\{|\pi_b(f_2)| > \tfrac{\lambda}{2}\}\right|$$

$$\le \frac{\|\pi_b(f_1)\|_2^2}{\lambda^2} + \left|\bigcup_{I \in \mathcal{B}} I\right|.$$

$$\lesssim \frac{1}{\lambda} \|f\|_{L^1}.$$

Interpolation now implies π_b is bounded on L^p for $p \in (1, 2]$.

For $p \in [2, \infty)$

π_b^* is similarly bounded from L^1 to weak-L^1, so $\|\pi_b^*(f)\|_p \lesssim \|b\|_{BMO} \|f\|_p$ for $p \in (1, 2]$. Thus, by duality,

$$\|\pi_b(f)\|_q \lesssim \|b\|_{BMO} \|f\|_q \quad \text{for } q \in [2, \infty).$$

$\square$.

We are now ready to prove $T(1)$

Thm: (T(1) Theorem).

Let T be a singular integral operator such that $T, T^*: \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R}) \cap L^1_{loc}(\mathbb{R})$.

Assume that

(1) $\max\left(|\langle h_I, T^*\varphi\rangle|, |\langle h_I, T\varphi\rangle| \right) \leq \|\varphi\|_2$

for all $\varphi \in \mathcal{S}(\mathbb{R})$, $I \in \mathcal{D}$.

(2) $\exists C > 0$ s.t.

$$|\langle \chi_{[0,1]}, T^* a\rangle| + |\langle \chi_{[0,1]}, Ta\rangle| \leq C$$

for all smooth $H^1([0,1])$ atoms a.

The T is bounded on $L^2([0,1])$