Define

$$c_k := 2^{n_k} |I_k|$$

$$a_k := c_k^{-1} f_k.$$

Then

$$f = \sum c_k a_k \qquad \text{and}$$

$$\begin{aligned} \sum_k |c_k| &= \sum_k 2^{n_k} |I_k| \\ &\leq 4 \sum_k 2^{n_k} \left| \{ x \in I_k \mid Sf(x) \; 2^{n_k - 1} \} \right| \\ &\leq 4 \sum_{k \in \mathbb{Z}^+} 2^k \left| \{ x \in [0,1] \mid Sf(x) > 2^{k-1} \} \right| \\ &\leq C \int_0^1 Sf \end{aligned}$$

$\square$.

<u>Prop</u>: If H is the Hilbert transform, then

$$\| Hf \|_{L^1([0,1])} \lesssim \| f \|_{\mathcal{H}^1([0,1])}.$$

Pf: $f = \sum c_j a_j$ where for each j,

- $\exists I_j$ s.t. $\mathrm{supp}(a_j) \subset I_j$.
- $\|a_j\|_2 |I_j|^{1/2} \le 1$.
- $\int_0^1 a_j = 0$

Recall that $\|Ha_j\|_{L^2} \le 1$ for all j.

$$\Rightarrow \quad \|Hf\|_{L^2} \le \sum |c_j| \le \|f\|_{\mathcal{H}^1([0,1])} \quad \square.$$

Lemma:

Let $F \in L^2([0,1])$ be such that $|\langle f, a\rangle| \le 1$ for all smooth $\mathcal{H}^1$-atoms a. Then $\|F\|_{BMO} \le 1$.

Pf: Fix $I \in \mathcal{D}$.

Let $a = \lambda |I|^{-1}(f - f_I)\chi_I$ where $\lambda > 0$ is chosen such that

$$1 = \|a\|_2 |I|^{1/2} = \left(\lambda^2 |I|^{-2} \int_I |f - f_I|^2\right)^{1/2} |I|^{1/2}$$

$$= \lambda \left(⨍_I |f - f_I|^2 \right)^{1/2}$$

Then a is an $\mathcal{H}^1$-atom and

$$\langle f, a \rangle = \langle f - f_I, a \rangle$$

$$= \lambda |I|^{-1} \int_I |f - f_I|^2$$

$$\Rightarrow \quad \lambda \fint_I |f - f_I|^2 \leq 1.$$

and $$\lambda^2 \fint_I |f - f_I|^2 = 1.$$

$\Rightarrow$ $\lambda \leq 1$ and thus

$$\fint_I |f - f_I|^2 \leq \frac{1}{\lambda} \leq 1 \qquad \square.$$

The $T(1)$ Theorem on $[0,1]$.

Def. A singular intergral operator, T, with kernel, K, is any linear operator,

$$T : \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$$

with the property that

$$\langle Tf, g \rangle = \int_{\mathbb{R}^2} K(x,y) f(x) g(y) \, dx \, dy$$

for all $f, g \in \mathcal{S}(\mathbb{R})$ with disjoint supports.

and where $K : \mathbb{R}^2 \to \mathbb{R}$ is a measurable function which is locally bounded on $\mathbb{R}^2 \setminus \{x=y\}$ and which satisfies

$$|K(x,y) - K(x',y)| \le \frac{|x-x'|^\delta}{|x-y|^{1+\delta}} \quad \forall\, |x-y| > 2|x-x'|$$

$$|K(x,y) - K(x,y')| \le \frac{|y-y'|^\delta}{|x-y|^{1+\delta}} \quad \forall\, |x-y| > 2|y-y'|$$

where $\delta \in (0,1]$ is fixed.

The decay estimates are enough to perform the Calderón-Zygmund argument demonstrating weak-L^1 boundedness. But how do we show L^2-boundedness for a non-convolution operator?

Our strategy so far has been to show that convolution-type SIOs are Fourier multiplier operators with bounded multiplier. This won't be possible with general SIO's.

This is where the $T(1)$ theorem comes into play.

Thm: (T(1) Theorem).

Let T be a singular integral operator such that $T, T^*: \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R}) \cap L^1_{loc}(\mathbb{R})$.

Assume that

(1) $\max(|\langle h_I, T^*\varphi\rangle|, |\langle h_I, T\varphi\rangle|) \leq \|\varphi\|_2$

for all $\varphi \in \mathcal{S}(\mathbb{R})$, $I \in \mathcal{D}$.

(2) $\exists C > 0$ s.t.

$$|\langle \chi_{[0,1]}, T^* a\rangle| + |\langle \chi_{[0,1]}, Ta\rangle| \leq C$$

for all smooth $\mathcal{H}^1([0,1])$ atoms a.

The T is bounded on $L^2([0,1])$

Notes:

Condition (1) can be understood as

$\|Th_I\|_2$, $\|T^*h_I\|_2 \leq \|h_I\|_2 = 1$ for all $I \in \mathcal{D}$

Condition (2) can be understood as

$T(1), T^*(1) \in BMO$ since the smooth

$\mathcal{H}^1$-atoms norm BMO

(Why is $\|Th_I\|_{L^2} \le 1$ not enough?. Consider $Th_I = \chi_{[0,1]}$ for all $I \in \mathcal{D}$).

Components of the $T(1)$ theorem

1. "Carleson measures"
2. Paraproducts
3. Cotlar's Lemma.

The Carleson condition

Def: A function $a : \mathcal{D} \to (0, \infty)$ is said to satisfy Carleson's condition if and only if for all $I \in \mathcal{D}$

$$\sum_{J \subset I} a(J) \le |I|$$

Prop: Let $a, b : \mathcal{D} \to (0,\infty)$, with a satisfying Carleson's condition. Then

$$\sum_{J \in \mathcal{D}} a(J)\, b(J) \leq 4 \int_0^1 \sup_{I \ni x} b(I)\, dx.$$

Note: This should remind us of the proof of
$|\langle f, g\rangle| \leq \|f\|_{H^1} \|g\|_{BMO}$.

pf: Suppose $\exists N \in \mathbb{Z}_+$ s.t.
$b(I), a(I) = 0$ for $|I| = 2^{-m}$ where $m > N$.

Define

$$A(I\,|\,x) := \sum_{x \in J \subset I} \frac{a(J)}{|J|} \qquad \forall x \in [0,1],\ I \in \mathcal{D}.$$

Let $I(x)$ be the maximal interval satisfying

$$A(I\,|\,x) \leq 2 \quad \text{and} \quad x \in I.$$

Claim: $|\{x \in J \mid I(x) \supset J\}| \geq \frac{1}{2}|J| \quad \forall J \in \mathcal{D}$

pf: Exercise.

Thus,

$$\sum_{J\in\mathcal{D}} a(J)b(J) = \sum_{J\in\mathcal{D}} |J|^{-1}\int_J a(J)b(J)$$

$$\leq 2\sum_{J\in\mathcal{D}} |J|^{-1}\int_J \chi_{\{x\in J \mid I(x)\supset J\}}\, a(J)b(J)$$

$$= 2\int_0^1 \sum_{x\in J\subset I(x)} |J|^{-1} a(J)b(J)$$

$$\leq 2\int_0^1 \sup_{J\ni x} b(J) \sum_{x\in J\subset I(x)} |J|^{-1} a(J)$$

$$= 2\int_0^1 \sup_{J\ni x} b(J)\; A(I(x)|x)$$

$$\leq 2\int_0^1 \sup_{J\ni x} b(J)$$

$\square$