Atomic Decomposition of dyadic $\mathcal{H}^1$

Def: $a : [0,1] \to \mathbb{C}$ is an $\mathcal{H}^1$ atom iff for some $I \in \mathcal{D}$ we have

- $\operatorname{supp}(a) \subset I$
- $\|a\|_{L^2([0,1])} \cdot |I|^{1/2} \leq 1.$
- $\int_0^1 a = 0$

The L^1-normalized Haar functions

$$|I|^{-1/2} h_I = \frac{1}{|I|}\left(\chi_{I_\ell} - \chi_{I_r}\right)$$

are a good example of $\mathcal{H}^1$ atoms.

Lemma: If a is an atom, then
$\|a\|_{\mathcal{H}^1} \leq 1$ and $\|a\|_{L^1} \leq 1$.

pf:

Let $a = \sum_{J \subset I} c_J h_J$

Then

$$\|Sa\|_2^2 = \int_I (Sa)^2(x)\,dx = \sum_{J\subset I} |c_J|^2 = \|a\|_2^2 \le |I|^{-1}$$

and Jensen's inequality implies

$$\|Sa\|_1 \le |I|^{1/2}\|Sa\|_2 = |I|^{1/2}\|a\|_2^2 \le 1.$$

$$\|a\|_1 \le |I|^{1/2}\|Sa\|_2 = |I|^{1/2}\|a\|_2^2 \le 1$$

$\square$

<u>Thm:</u> For every $f \in \mathcal{H}^1([0,1])$ there exists a sequence of scalars $\{c_j\}_{j=1}^{\infty}$ and atoms $\{a_j\}_{j=1}^{\infty}$ such that $f = \sum c_j a_j$ converges in $\mathcal{H}^1$ with $\sum_j |c_j| \le C\|f\|_{\mathcal{H}^1}$ for some absolute constant C

<u>Pf</u>: Let $f = \sum_I b_I h_I$ and assume $\|Sf\|_2 = 1$. Also, assume f has a finite Haar expansion. For each $I \in \mathcal{D}$, define λ_I by

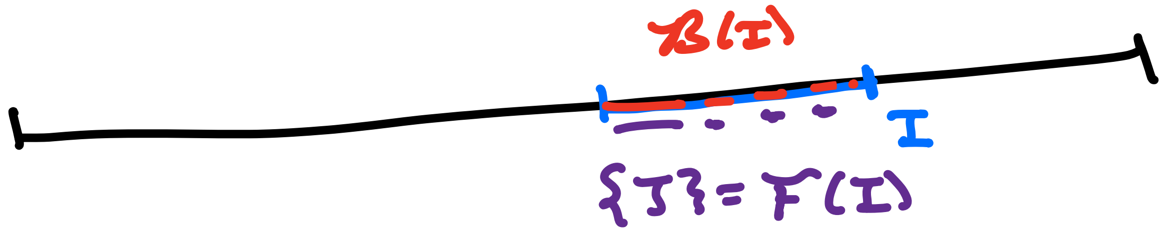
$$\lambda_I := \min\{2^k \mid k \in \mathbb{Z}_+,\ |\{x \in I \mid (Sf)(x) > 2^k\}| \le \tfrac{1}{4}\}$$

Denote

$$\mathcal{B}(I) := \{ x \in I \mid (SF)(x) > \lambda_I \}.$$

and

$$\mathcal{F}(I) := \text{maximal dyadic intervals in } \mathcal{B}(I).$$



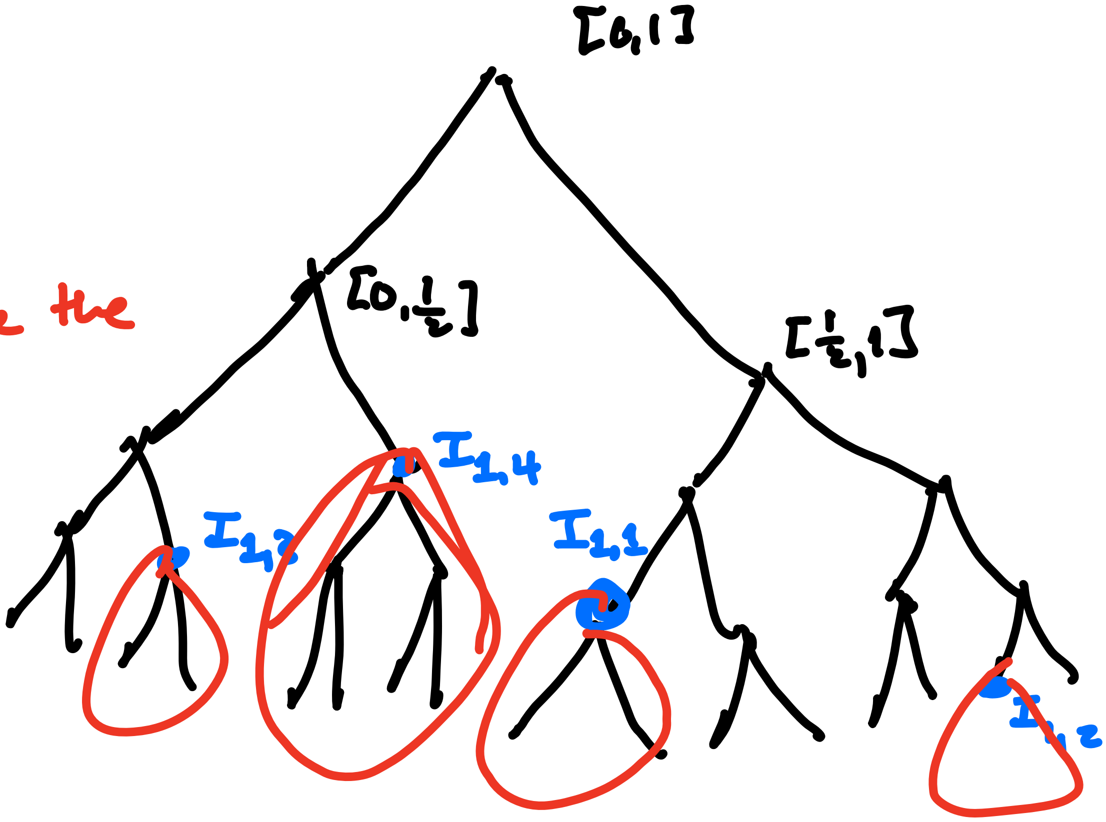
We now perform an iterative decomposition of the dyadic intervals, $\mathcal{D}$.

Let $\lambda_{[0,1]}$, $\mathcal{B}([0,1])$ and $\mathcal{F}([0,1])$ be defined as above.

If $\mathcal{F}([0,1]) = \phi$, then $\mathcal{B}([0,1]) = \phi$ and we are done.

If not, denote $\{I_{2,j}\}_j := \mathcal{F}([0,1])$.

Prune the tree



Let $T(I_{1,j}) := \{I \in \mathcal{D} \mid I \subseteq I_{1,j}\}$ be the subtree with root $I_{1,j}$.

Denote $\Upsilon([0,1])$ as the "pruned" tree with maximal node (root) $[0,1]$. I.e let

$$\Upsilon([0,1]) := \mathcal{D} \setminus \left(\bigcup_j T(I_{1,j}) \right)$$

f restricted to the pruned tree is now bounded by $\lambda([0,1])$. I.e. if

$$f_{[0,1]} := \sum_{I \in T([0,1])} b_I h_I$$

Then

$$S f_{[0,1]} \leq \lambda([0,1]),$$

and in particular

$$\| S f_{[0,1]} \|_2^2 = \| f_{[0,1]} \|_2^2 = \sum_{I \in \mathcal{T}([0,1])} b_I^2 \leq \lambda^2([0,1]).$$

$f_{[0,1]}$ will be (up to a factor) our first atom.

We repeat this process with each tree pruned away in the previous step.

For each $I_{1,j}$, define $\lambda(I_{1,j})$, $\mathcal{B}(I_{1,j})$ and $\mathcal{F}(I_{1,j})$ as above and prune the tree with root $I_{1,j}$

$$\mathcal{T}(I_{1,j}) := T(I_{1,j}) \setminus \Big(\bigcup_{J \in \mathcal{F}(I_{1,j})} T(J) \Big).$$

As before,

$$f_{I_{1,j}} := \sum_{I \in \Upsilon(I_{1,j})} b_I h_I .$$

Then $\quad S f_{I_{1,j}} \leq \lambda(I_{1,j})$

and

$$\| S f_{I_{1,j}} \|_2^2 = \| f_{I_{1,j}} \|_2^2 \leq \lambda^2(I_{1,j}) \, |I_{1,j}| .$$

Continuing in this fashion, we obtain a collection of disjoint subtrees, $\{\Upsilon_k\}_k$, with roots $I_k \in \mathcal{D}$ and $f_k = \sum_{I \in \Upsilon_k} b_I h_I$ with

$$\| f_k \|_2 \leq 2^{n_k} |I_k|^{1/2}$$

The minimality of n_k, implies

$$\frac{1}{4} |I_k| \leq | \{ x \in I_k \mid Sf(x) > 2^{n_k - 1} \} | .$$

Also, if $I_k \subsetneq I_{k'}$ for $k \neq k'$, then $n_k > n_{k'}$

Define

$$c_k := 2^{n_k} |I_k|$$

$$a_k := c_k^{-1} f_k.$$

Then

$$f = \sum c_k a_k \qquad \text{and}$$

$$\sum_k |c_k| = \sum_k 2^{n_k} |I_k|$$

$$\leq 4 \sum_k 2^{n_k} |\{x \in I_k \mid Sf(x) > 2^{n_k - 1}\}|$$

$$\leq 4 \sum_{k \in \mathbb{Z}^+} 2^k |\{x \in [0,1] \mid Sf(x) > 2^{k-1}\}|$$

$$\leq C \int_0^1 Sf \qquad \square.$$

<u>Prop</u>: If H is the Hilbert transform, then

$$\| Hf \|_{L^1([0,1])} \leq \| f \|_{\mathcal{H}^1([0,1])}.$$
