We've shown

$$\mathrm{BMO} \subset (\mathcal{H}^1)^*$$

Now we begin to show the reverse inclusion.

Prop: Given $f \in \mathcal{H}^1$ with finite Haar expansion, there exists $g \in \mathrm{BMO}$ with $\|g\|_{\mathrm{BMO}} = 1$ and

$$|\langle f, g\rangle| \geq c \|f\|_{\mathcal{H}^1}$$

where $c > 0$ is an absolute constant.

pf:

Let $f = \sum a_I h_I$ and for $t \in [0,1]$,

define $f_t = \sum r_I(t) a_I h_I$

By Khinchine inequality

$$\mathbb{E}_t \left[\|f_t\|_{L^1} \right] \geq c \, \|Sf\|_{L^1}.$$

$\Rightarrow \exists t_0 \in [0,1]$ s.t. $\|f_{t_0}\|_{L^1} \geq \frac{c}{2} \|Sf\|_{L^1}$.

Since $(L^1)^* = L^\infty$, $\exists g \in L^\infty([0,1])$ s.t.

$$\langle f_{t_0}, g \rangle = \|f_{t_0}\|_{L^1}$$

If $g = \sum_I c_I h_I$, then let

$$g_{t_0} = \sum r_I(t_0) c_I h_I$$

Then $\int_0^1 g_{t_0} = 0$ and

$$\|g_{t_0}\|_{BMO} = \sup_{I \in \mathcal{D}} |I|^{-1} \sum_{J \subset I} |r_J(t_0) c_J|^2$$

$$= \sup_I |I| \sum_{J \subset I} |c_J|^2$$

$$= \|g\|_{BMO} \leq 2\|g\|_\infty .$$

$\Rightarrow$

$$\langle f, g_{t_0} \rangle = \langle f_{t_0}, g \rangle = \|f_{t_0}\|_{L^2} \geq \frac{c}{2} \|Sf\|_{L^1} \quad \square .$$

<u>Thm</u>: For all $L \in \left(\mathcal{H}^1([0,1])\right)^*$, there exists a unique $g \in BMO$ with $\|g\|_{BMO} = \|L\|$ s.t.

$$L(f) = \langle f, g \rangle$$

<u>for all</u> $f \in \mathcal{H}^1([0,1])$ with finite Haar expansion.

<u>Pf</u>.

Let $f \in \mathcal{H}^1([0,1])$ and for $n \in \mathbb{Z}_+ \cup \{0\}$,

$$f_n := \sum_{I \in \mathcal{D}_n} \langle f, h_I \rangle h_I .$$

Then
$$Sf = \left(\sum_{n=0}^{\infty} f_n^2 \right)^{1/2} \in L^1([0,1])$$

The map $f \mapsto \{f_n\}_{n=0}^{\infty}$ embeds $\mathcal{H}^1$ into $L^1([0,1]; \ell^2)$ isometrically.

If $L \in (\mathcal{H}^1)^*$, then we can extend L to $\hat{L} \in \left(L^1([0,1]; \ell^2)\right)^*$
$$= L^\infty([0,1]; \ell^2).$$
by Hahn-Banach.

Thus, $\exists \{g_n\}_{n=0}^{\infty}$ s.t.
$$\sup_x \sum |g_n(x)|^2 = \|\hat{L}\| = \|L\|.$$

and
$$\hat{L}(\{\varphi_n\}) = \sum_{n=0}^{\infty} \int_0^1 \varphi_n(x)\, g_n(x)$$

for $\{\varphi_n\} \in L^1([0,1]; \ell^2)$

And for $f \in \mathcal{H}^1$,

$$L(f) = \tilde{L}(f) = \sum_{n=0}^{\infty} \int_0^1 f_n(x)\, g_n(x)$$

$$= \sum_{n=0}^{\infty} \sum_{I \in \mathcal{D}_n} \int_0^1 \langle f, h_I \rangle h_I \, g_n$$

$$= \sum_{n=0}^{\infty} \sum_{I \in \mathcal{D}} \langle f, h_I \rangle \langle g_n, h_I \rangle .$$

$$= \langle f, g \rangle$$

where

$$g = \sum_{n=0}^{\infty} \sum_{I \in \mathcal{D}_n} \langle g_n, h_I \rangle h_I .$$

Now

$$\|g\|_{BMO} = \sup_{J \in \mathcal{D}} |J|^{-1} \sum_{n=0}^{\infty} \sum_{I \subset J} |\langle h_I, g_n \rangle|^2$$

$$\leq \sup_{J \in \mathcal{D}} |J|^{-1} \int_J \sum |g_n|^2$$

$$\leq \|\{g_n\}\|_{L^\infty([0,1]; \ell^2)} = \|L\|$$

$\square$.