Properties of the Fourier Transform on $\mathbb{R}$

- Let $\mu \in M(\mathbb{R})$, let τ_y denote translation by $y \in \mathbb{R}$,

$$(\tau_y \mu)(E) = \mu(E-y)$$

Then

$$\widehat{\tau_y \mu}(\xi) = e(-\xi y)\,\hat{\mu}(\xi) \quad \forall \xi \in \mathbb{R}$$

- Let $f, g \in L^1(\mathbb{R})$, and

$$(f*g)(x) := \int_{\mathbb{R}} f(x-y)\, g(y)\, dy.$$

Then $f*g \in L^1(\mathbb{R})$ and

$$\widehat{f*g} = \hat{f}\,\hat{g}$$

- For $\lambda > 0$, let $m_\lambda(x) = \lambda x$

  $f \in L^1(\mathbb{R})$, then

$$\widehat{f \circ m_\lambda}(\xi) = \lambda^{-1} \left(\hat{f} \circ m_{\lambda^{-1}} \right)(\xi).$$

$$= \lambda^{-1} \hat{f}(\xi/\lambda).$$

- $$\widehat{\left(\frac{d}{dx}\right)^\alpha f}(\xi) = (2\pi i)^\alpha \xi^\alpha \hat{f}(\xi)$$

$$\left(\frac{d}{dx}\right)^\beta \hat{f}(\xi) = (-2\pi i)^\beta \widehat{x^\beta f}(\xi).$$

Therefore, if $f \in \mathcal{S}(\mathbb{R})$ then
$\hat{f} \in \mathcal{S}(\mathbb{R})$.

- Let Φ be the Gaussian

$\Phi(x) := e^{-\pi x^2}$ then

$\hat{\Phi}(\xi) = \Phi(\xi)$.

Let's show this:

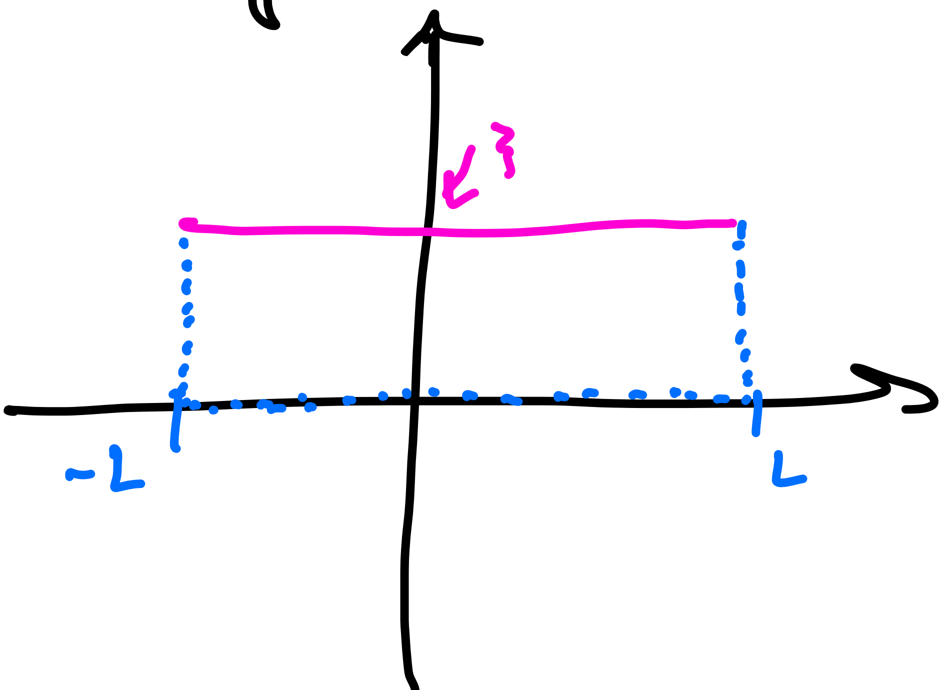
$$\hat{\Phi}(\xi) = \int_{\mathbb{R}} e(-x\xi)\, e^{-\pi x^2}\, dx$$

$$= \int_{\mathbb{R}} e^{-2\pi i x\xi - \pi x^2}\, dx$$

$$= e^{-\pi \xi^2} \int_{\mathbb{R}} e^{-\pi (x - i\xi)^2}\, dx$$

$$= \lim_{L \to \infty} e^{-\pi \xi^2} \int_{-L}^{L} e^{-\pi (x - i\xi)^2}\, dx$$

Contour Integration:



$$\int_{-L}^{L} e^{-\pi (x - i\xi)^2} = \int_{\xi}^{0} e^{-\pi(-L - iy)^2}\, dy + \int_{0}^{\xi} e^{-\pi(L - iy)}\, dy$$

$$+ \int_{-L}^{L} e^{-\pi x^2}$$

Note that

$$\left| \int_{\xi}^{0} e^{-\pi(-L-iy)^2}\, dy + \int_{0}^{\xi} e^{-\pi(L-iy)}\, dy \right|$$

$$\lesssim e^{-L^2\pi} \int_0^{\xi} e^{y^2}\, dy \xrightarrow{L\to\infty} 0.$$

Thus

$$\lim_{L\to\infty} e^{-\pi\xi^2} \int_{-L}^{L} e^{-\pi(x-i\xi)^2} = e^{-\pi\xi^2} \lim_{L\to\infty} \int_{-L}^{L} e^{-\pi x^2}\, dx$$

$$= e^{-\pi\xi^2}.$$

- Fourier Inversion Formula

$$f(x) = \int_{\mathbb{R}} e(x\xi)\hat{f}(\xi)\, d\xi.$$

$$\int e(x\xi)\hat{f}(\xi)\, d\xi = \lim_{\varepsilon\to 0} \int_{\mathbb{R}} e(x\xi)\, e^{-\pi\varepsilon^2\xi^2} \hat{f}(\xi) d\xi$$

$$= \lim_{\varepsilon\to 0} \int e(x\xi)\, e^{-\pi\varepsilon^2\xi^2} \int e(-y\xi) f(y)\, dy\, d\xi$$

$$= \lim_{\varepsilon\to 0} \int \left[\int e(\xi(x-y))\, e^{-\pi\varepsilon^2\xi^2} d\xi \right] f(y)\, dy$$

$$= \lim_{\varepsilon \to 0} \int \varepsilon^{-1} e^{-\pi \varepsilon^{-2}(x-y)^2} f(y)\, dy.$$

$$= \lim_{\varepsilon \to 0} \int \frac{1}{\varepsilon} \left(\Phi \circ m_{\varepsilon^{-1}} \right)(x-y)\, f(y)\, dy.$$

$$\boxed{\text{Claim: } \left\{ \frac{1}{\varepsilon} \left(\Phi \circ m_{\varepsilon^{-1}} \right) \right\}_{\varepsilon > 0} \text{ is an approximate identity.}}$$

Therefore,

$$\int e(x\xi) \hat{f}(\xi) = \lim_{\varepsilon \to 0} \left(\Gamma_\varepsilon * f \right)(x)$$

where $\Gamma_\varepsilon := \frac{1}{\varepsilon} \left(\Phi \circ m_{\varepsilon^{-1}} \right)$

Thus, if $f \in \mathcal{S}$, then

$$f(x) = \lim_{\varepsilon \to 0} (\Gamma_\varepsilon * f)(x) = \int_{\mathbb{R}} e(x\xi) \hat{f}(\xi)\, d\xi$$

- Plancherel Theorem

If $f \in \mathcal{S}(\mathbb{R})$, $\|\hat{f}\|_2 = \|f\|$.

Pf.

By Fubini,

$$\langle f, \hat{g} \rangle_{L^2(\mathbb{R})} := \int_{\mathbb{R}} f(x) \overline{\hat{g}(x)}\, dx$$

$$= \int_{\mathbb{R}} \check{f}(\xi) \overline{g(\xi)}\, d\xi$$

$$= \langle \check{f}, g \rangle_{L^2(\mathbb{R})}$$

where $\check{f}(\xi) := \int f(x) e(x\xi)\, dx$

Therefore,

$$\langle \hat{f}, \hat{g} \rangle_{L^2(\xi)} = \langle f, g \rangle_{L^2(x)}$$

$$\Rightarrow \quad \|\hat{f}\|_{L^2}^2 = \|f\|_{L^2}^2. \qquad \square.$$

<u>Cor</u>: If $f \in L^1$, $\hat{f} \equiv 0$. Then $f = 0$.

Now that we know that the Fourier transform is unique, we return to the question of in what sense $f(x) = \int e(x\xi)\hat{f}(\xi)\,d\xi$ when we don't have the benefit of assuming $f \in \mathcal{S}(\mathbb{R})$.

For $f \in L^1(\mathbb{R})$, $T \in (0,\infty)$, define

$$S_T f(x) := \int_{-T}^{T} e(x\xi)\hat{f}(\xi)\,d\xi.$$

Since $f \in L^1 \Rightarrow \hat{f} \in L^\infty$, $S_T f$ makes sense.

Just as in the proof of Fourier inversion, we can think of S_T as a convolution operator.

$$S_T f(x) = \int_{-T}^{T} e(x\xi)\hat{f}(\xi)\,d\xi$$

$$= \int_{-T}^{T} e(x\xi) \int_{\mathbb{R}} e(-y\xi) f(y)\,dy\,d\xi$$

$$= \int_{\mathbb{R}} \left(\int_{-T}^{T} e((x-y)\xi)\,d\xi \right) f(y)\,dy.$$

Let $D_T(t) := \int_{-T}^{T} e(t\xi)\,d\xi$.

Then

$$S_T f(x) = \int_{\mathbb{R}} D_T(x-y) f(y)\,dy$$

$$= (D_T * f)(x)$$

D_T is known as the Dirichlet Kernel.