Def: (Dyadic Square Function)

For any $f \in L^1([0,1])$ with $\int_0^1 f = 0$ we define

$$Sf(x) = \left(\sum_{I \in \mathcal{D}} |a_I|^2 h_I^2(x) \right)^{1/2}.$$

for $f = \sum_{I \in \mathcal{D}} a_I h_I$

Def: (Dyadic BMO and Dyadic Hardy)

The dyadic Hardy space is defined as

$$\mathcal{H}^1([0,1]) = \left\{ f \in L^1([0,1]) \,\middle|\, \begin{array}{l} \int_0^1 f = 0 \\ Sf \in L^1([0,1]) \end{array} \right\}.$$

The dyadic BMO space is defined as the space of all $f \in L^2([0,1])$ with

$$\|f\|_{BMO} := \sup_{I \in \mathcal{D}} \left(\fint_I |f(x) - f_I|^2 \, dx \right)^{1/2} < \infty.$$

Note: If $f = \sum_{I \in \mathcal{D}} a_I h_I$, then

$$\|f\|_{BMO} = \sup_{I \in \mathcal{D}} |I|^{-1} \sum_{\substack{J \subset I \\ J \in \mathcal{D}}} |a_J|^2$$

Thm: (Littlewood - Paley for dyadic square function)

For $f \in L^p([0,1])$, $p \in (1,\infty)$ and $\int f = 0$ one has

$$c_p^{-1} \|f\|_p \leq \|Sf\|_p \leq c_p \|f\|_p .$$

---

pf: Let $a_I := \langle f, h_I \rangle$

Khinchine's inequality implies that if one indexes the rademacher sequence by $I \in \mathcal{D}$, $\{r_I\}_{I \in \mathcal{D}}$, then

$\exists\, C = C(p)$ s.t. for all $x \in [0,1]$

$$C^{-1}\left(\sum |a_I|^2 h_I^2(x)\right)^{p/2}$$

$$\leq \mathbb{E}_t \left|\sum r_I(t) a_I h_I(x)\right|^p$$

$$\leq C \left(\sum |a_I|^2 h_I^2(x)\right)^{p/2}.$$

then, since $\sup_I \| r_I \|_\infty \leq 1$,

$$\mathbb{E} \left\| \sum_I r_I(t) a_I h_I \right\|_p^p \lesssim \| F \|_p^p$$

By the Multiplier thm.

A duality argument completes the argument $\square$.

Since $Sf(x) \sim \mathbb{E}_t \left|\sum r_I(t) a_I h_I(x)\right|$

$Sf \in L^1$ in some sense says that for generic Haar multiplier operators, M_t, $M_t f \in L^1$.

$\mathcal{H}^1$ - BMO Duality

<u>Thm:</u> Let f be a finite linear combination of Haar functions. For any $g \in BMO$, one has

$$|\langle f, g \rangle_{L^2([0,1])}| \lesssim \|f\|_{\mathcal{H}^1} \|g\|_{BMO}.$$

<u>pf.</u>

Let $g = \sum b_J h_J$ and $f = \sum a_J h_J$.

with $a_J, b_J \in \mathbb{R}$ for all $J \in \mathcal{D}$.

Since $\|g\|_{BMO} = \sup_{I \in \mathcal{D}} |I|^{-1} \sum_{J \subset I} |b_J|^2$, it suffices to assume that g has a finite Haar expansion

Define

$$S(g|I)(x) := \left(\sum_{J \subset I} b_J^2 h_J^2(x) \right)^{1/2}$$

<u>Example:</u>

$$g = \sum_{k=1}^{N} 2^{-k/2} h_{[0, 2^{-k})}$$

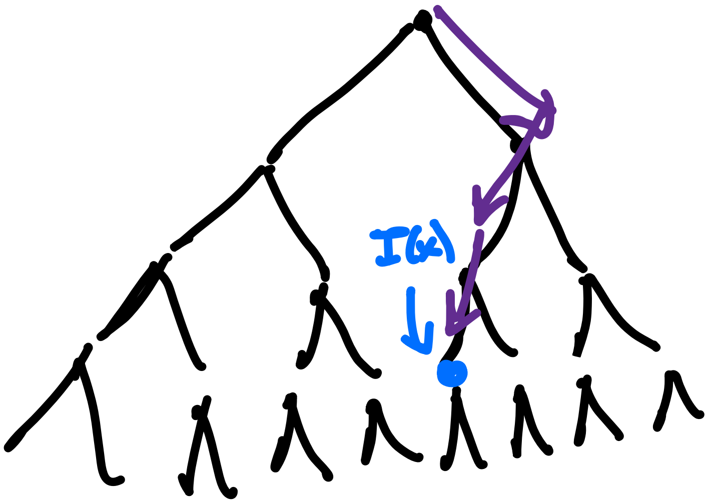
Let $I(x) =$ largest interval s.t.

$$S^2(g|I)(x) \leq 2 \sup_{J \ni x} \fint S^2(g|J)$$

$$= 2 \sup_{J \ni x} \fint_J |g - g_J|^2$$

$$= 2 \sup_{J \ni x} |J|^{-1} \sum_{H \subset I} b_H^2$$

$$\le 2 \|g\|_{BMO}$$



For each $x \in [0,1]$, such a $I(x)$ exists since g has a finite Haar expansion (Eventually, $g = g_J$ for J small enough).

Claim: For each $J \in \mathcal{D}$,

$$|\{x \in J \mid I(x) \supset J\}| \ge \frac{1}{2}|J|$$

To prove the claim, let

$$A := \{x \in J \mid I(x) \subsetneq J\}|$$

For $x \in J$, the maximality of $I(x)$ implies

$$S(g|J)(x) > 2 \sup_{L \ni x} \fint_L S(g|L) \geq 2 \fint_J S(g|J)$$

Thus

$$\int_J S(g|J)(x) \geq \int_A S(g|J)(x)$$

$$> 2 \int_A \fint_J S(g|J)$$

$$= 2|A|\,|J|^{-1} \int_J S(g|J)$$

$$\Rightarrow \quad |A| < \tfrac{1}{2}|J|$$

Now we perform a careful Cauchy-Schwarz argument.

$$|\langle f,g\rangle| \leq \sum_{J \in D} |a_J b_J| = \sum_{J \in D} |J|^{-1} \int_0^1 \chi_J |a_J b_J|$$

$$\leq 2 \sum_J \int_0^1 |J|^{-1} \chi_J \chi_{\{x \in J \mid I(x) \supset J\}} |a_J b_J|$$

$$= 2 \int_0^1 \sum_{J \subset I(x)} h_J^2 |a_J b_J|$$

$$\le 2\int_0^1 \Big(\sum_J |a_J|^2 h_J^2\Big)^{1/2} \Big(\sum_{J \subset I(x)} |b_J|^2 h_J^2\Big)^{1/2}.$$

$$= 2\int_0^1 Sf(x)\, S(g|I(x))(x)\, dx$$

$$\le 2\, \|Sf\|_{L^1} \sup_{x\in[0,1]} |S(g|I(x))(x)|$$

$$\le 4\, \|f\|_{\mathcal{H}^1} \|g\|_{BMO}.$$

We've shown

$$BMO \subset (\mathcal{H}^1)^*$$

Now we begin to show the reverse inclusion.

Prop: Given $f \in \mathcal{H}^1$ with finite Haar expansion, there exists $g \in BMO$ with $\|g\|_{BMO} = 1$ and

$$|\langle f, g\rangle| \ge c\, \|f\|_{\mathcal{H}^1}$$

where $c > 0$ is an absolute constant.