Dyadic BMO and the Hardy Space.

BMO and the John-Nirenberg inequality

We define BMO on $[0,1]$ with respect to dyadic intervals.

$$f^{\#}(x) = \sup_{x \in Q \in \mathcal{D}} \fint_Q |f(y) - f_Q|$$

$$f \in BMO_{dyad}([0,1]) \text{ iff } f^{\#} \in L^\infty.$$

<u>Thm:</u> (John-Nirenberg Inequality)

Let $f \in BMO_{dyad}([0,1])$. Then for any $Q \in \mathcal{D}$

$$\left| \{ x \in Q \mid |f(x) - f_Q| > \lambda \} \right| \leq C|Q| \exp\left(-c \frac{\lambda}{\|f\|_{BMO}}\right).$$

For some universal $c, C > 0$.

Pf:

It suffices to prove that for $M \in \mathbb{Z}_+$,

$$\left|\{x \in Q \mid |f(x) - f_Q| > 10M \|f\|_{BMO}\}\right| \leq C|Q| 2^{-M}$$

We will induct on $m = 1, 2, \dots, M$.
For the base case. For any $Q \in \mathcal{D}$

$$\left|\{x \in Q \mid |f(x) - f_Q| > 10 \|f\|_{BMO}\}\right|$$

$$\leq |Q|$$

Suppose that for each $Q \in \mathcal{D}$

$$\left|\{x \in Q \mid |f(x) - f_Q| > 10m \|f\|_{BMO}\}\right|$$

$$\leq C|Q| 2^{-m}.$$

Fix $Q \in \mathcal{D}$ and perform a C-Z decomposition at height $2\|f\|_{BMO}$, with $f - f_Q = g + b = \tilde{f}$

and

$$b = \sum_{Q' \in \mathcal{B}} \chi_{Q'} (\tilde{f} - \tilde{f}_{Q'})$$

$$g = \tilde{f} - \sum_{Q' \in \mathcal{B}} \chi_{Q'} (\tilde{f} - \tilde{f}_{Q'})$$

Then

$$\sum |Q'| \le \frac{\int_Q |\tilde{f}|}{2\|\tilde{f}\|_{BMO}} \le \frac{|Q|}{2}.$$

and $$|g| \le 4\|\hat{\tilde{f}}\|_{BMO} \le 10\|f\|_{BMO}.$$

Then

$$|\{x \in Q \mid |f(x) - f_Q| > 10(m+1)\|f\|_{BMO}\}|$$

$$= |\{x \in Q \mid |g+b| > 10(m+1)\|f\|_{BMO}\}|$$

$$\le |\{x \in Q \mid |b| > 10m\|f\|_{BMO}\}|$$

$$= \sum_{Q' \in \mathcal{B}} |\{x \in Q' \mid |b| > 10m\|f\|_{BMO}\}|$$

$$= \sum_{Q' \in \mathcal{B}} |\{x \in Q' \mid |\tilde{f}(x) - f_{Q'}| > 10m\|f\|_{BMO}\}|$$

$$\le \sum |Q'| 2^{-m}$$

$$\le \frac{|Q|}{2} 2^{-m}.$$

$\square$.

Corollary:

For every $p \in [1,\infty)$ $\exists\, C = C(p)$ s.t.

$$\sup_{Q \in \mathcal{D}} \left(⨍ |f - f_Q|^p \right)^{1/p} \le C \left[\sup_{Q \in \mathcal{D}} \left(⨍ |f - f_Q| \right) \right]$$

Pf:

Fix $Q \in \mathcal{D}$

Note that

$$\sup_Q \left(⨍ |f - f_Q| \right) = \|f\|_{BMO}.$$

Then for any $Q \in \mathcal{D}$

$$\frac{1}{|Q|} \int |f - f_Q|^p \, \|f\|_{BMO}^{-p}$$

$$\le \frac{1}{|Q|} \int_0^\infty p\lambda^{p-1} \left| \{ |f - f_Q| > \lambda \|f\|_{BMO} \} \right|$$

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$$\le \frac{1}{|Q|} \int_0^\infty p\lambda^{p-1} c|Q| \exp(-c\lambda)$$

$$\le C'(p)$$

$\square$.

Def: (Dyadic Square Function)

For any $f \in L^1([0,1])$ with $\int_0^1 f = 0$ we define

$$Sf(x) = \Big(\sum_{I \in \mathcal{D}} |a_I|^2 h_I^2(x) \Big)^{1/2}.$$

for $f = \sum_{I \in \mathcal{D}} a_I h_I$

Def: (Dyadic BMO and Dyadic Hardy)

The dyadic Hardy space is defined as

$$\mathcal{H}^1([0,1]) = \Big\{ f \in L^1([0,1]) \;\Big|\; \int_0^1 f = 0,\ Sf \in L^1([0,1]) \Big\}.$$

The dyadic BMO space is defined as the space of all $f \in L^2([0,1])$ with

$$\|f\|_{BMO} := \sup_{I \in \mathcal{D}} \Big(\fint_I |f(x) - f_I|^2 \, dx \Big)^{1/2} < \infty.$$