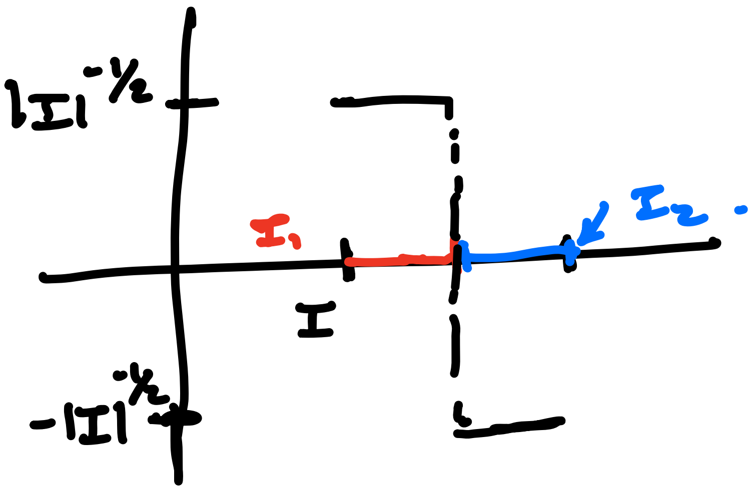
Atoms, Wave Packets, $\mathcal{H}^1$, BMO and Paraproducts

We've discussed that dyadic frequency decompositions are the most natural. In order to study these ideas further in a technically clean manner, we build the theory of Haar functions on the interval $[0,1]$.

Def: Let

$$\mathcal{D}_{\le n} = \bigcup_{k=0}^{n} \mathcal{D}_k = \bigcup_{k=0}^{n} \left\{ [j2^{-k},(j+1)2^{-k}) : j=0,\dots,2^k-1 \right\}$$

For $I \in \mathcal{D}_{\le\infty} =: \mathcal{D}$, I has two children $I_1 \cup I_2 = I$. Define $h_I := (\chi_{I_1} - \chi_{I_2})|I|^{-1/2}$



h_I is the Haar function associated to I the Haar systems

$$\{h_I\}_{I\in\mathcal{D}}.$$

Let $\Sigma_n :=$ σ-algebra generated by $\mathcal{D}_n$

and define

$$\mathbb{E}_n f := \mathbb{E}[f \mid \Sigma_n] \qquad \text{for } f\in L^1([0,1]) \text{ and } n\geq 0.$$

where $\mathbb{E}[\cdot\mid\cdot]$ is the conditional expectation operator. I.e.

$$\mathbb{E}_n(f) = \sum_{I\in\mathcal{D}_n} \chi_I \fint_I f$$

Thus for $f \in C([0,1])$

$$\lim_{n\to\infty} \| \mathbb{E}_n(f) - f \|_\infty = 0$$

and for $f \in L^p([0,1])$

$$\lim_{n\to\infty} \| \mathbb{E}_n f - f \|_p = 0 .$$

<u>Lemma</u> For each $n \geq 1$ one has

$$\mathbb{E}_n(f) = \int_0^1 f + \sum_{I \in \mathcal{D}_{\leq n-1}} \langle f, h_I \rangle h_I$$

and

$$\chi_J \fint_J f = \int_0^1 f + \sum_{\substack{I \supset J \\ I \in \mathcal{D}}} \langle f, h_I \rangle h_I$$

<u>Pf</u>: By induction,

we want to show that

$$E_n(f) = \mathbb{E}_{n-1}(f) + \sum_{I \in \mathcal{D}_{n-1}} \langle f, h_I \rangle h_I .$$

For $n=1$,

$$E_1(f) = \chi_{[0,\frac{1}{2})} \fint_{[0,\frac{1}{2})} f + \chi_{[\frac{1}{2},1)} \fint_{[\frac{1}{2},1)} f$$

$$= 2\chi_{[0,\frac{1}{2})} \int \chi_{[0,\frac{1}{2})} f + 2\chi_{[\frac{1}{2},1)} \int \chi_{[\frac{1}{2},1)} f$$

$$= \int_0^1 f + \chi_{[0,\frac{1}{2})} \int \left(\chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}\right) f$$

$$\qquad + \chi_{[\frac{1}{2},1)} \int \left(\chi_{[\frac{1}{2},1)} - \chi_{[0,\frac{1}{2})}\right) f$$

$$= \int_0^1 f + (h_{[0,1]}) \int h_{[0,1]} f$$

$$= \mathbb{E}_0(f) + \langle f, h_{[0,1]} \rangle h_{[0,1]}$$

In general,

$$\chi_{I_1} \fint_{I_1} f + \chi_{I_2} \fint_{I_2} f = \chi_I \fint_I f + \langle f, h_I \rangle h_I.$$

where $I = I_1 \cup I_2$.

$\square$

For $I \neq J$ $\langle h_I, h_J \rangle = 0$

Thus, $\{\chi_{[0,1]}\} \cup \{h_I\}_{I \in \mathcal{D}}$ is an orthonormal basis for $L^2([0,1])$.

Schauder

Relationship between Haar functions and singular integrals

Lemma:

Let T be a C-Z operator.

$$\|Th_I\|_1 \lesssim |I|^{1/2}$$

pf.

$$\int_{x \in 2I} |Th_I(x)| \lesssim \|Th_I\|_{L^2} |I|^{1/2}$$

$$\lesssim \|h_I\|_{L^2} |I|^{1/2} = |I|^{1/2}.$$

$$\int_{x \notin 2I} |Th_I(x)| \leq \int_I \int_{x \notin 2I} |K(x-y) - K(x-y_I)| |h_I(y)|$$

$$\lesssim \|h_I\|_1 \sim |I|^{1/2}$$

$\square$.

Haar Basis Multiplier operators

<u>Thm:</u> Let $\{x_I\}_{I\in\mathcal{D}}$ be an arbitrary sequence of scalars such that $|x_I| \le C$ for all $I \in \mathcal{D}$. Then the multiplier operator, defined on all functions $f \in L^1([0,1])$ with finite Haar expansion

$$Tf := \sum_{I\in\mathcal{D}} x_I \langle f, h_I\rangle h_I.$$

is weak-L^1 bounded and L^p bounded for $p \in (1,\infty)$.

<u>pf:</u>

First, the L^2 bound:

Since $\{1\} \cup \{h_I\}_{I\in\mathcal{D}}$ is an orthonormal basis for $L^2([0,1])$

$$\|Tf\|_2^2 = \sum_{I\in\mathcal{D}} |x_I|^2 |\langle f, h_I\rangle|^2 \le C^2 \sum_{I\in\mathcal{D}} |\langle f, h_I\rangle|^2$$

$\le C^2 \|f\|_{L^2}$

By interpolation and duality it suffices to demonstrate the weak-L^1 bound.

Goal: $|\{x \in [0,1] \mid |Tf(x)| > \lambda\}| \le C\lambda^{-1}\|f\|_1$ for all $\lambda > 0$.

Let $f = g + b$ be a Calderón-Zygmund decomposition at height λ.

Then

$$|\{|Tf| > \lambda\}| \le |\{|Tg| > \tfrac{\lambda}{2}\}| + |\{|Tb| > \tfrac{\lambda}{2}\}|$$

and

$$|\{|Tg| > \tfrac{\lambda}{2}\}| \le 4\frac{\|Tg\|_2^2}{\lambda^2} \le 4C^2 \frac{\|g\|_2^2}{\lambda^2} \le 4C^2 \frac{\|f\|_1}{\lambda}.$$

For b, note that $\operatorname{supp}(b) = \operatorname{supp}(Tb) = \bigcup_{Q \in \mathcal{B}} Q$

Thus,

$$|\{|Tb| > \tfrac{\lambda}{2}\}| \le \Big|\bigcup_{Q \in \mathcal{B}} Q\Big| \le \frac{\|f\|_1}{\lambda} \qquad \square.$$