$$\varphi(x) = \begin{cases} 0 & |x| > 2 \\ 1 & |x| \le 1. \end{cases}$$

Now define $\psi(x) = \varphi(x) - \varphi(2x)$

Then

$$\sum_{j=-M}^{M} \psi(2^{-j}x) = \varphi(2^{-M}x) - \varphi(2^{M}x)$$

$$= \begin{cases} 0, & |x| < 2^{-M},\ |x| > 2^{M+1} \\ 1, & 2^{-M+1} < |x| < 2^{M}. \end{cases}$$

$\square$.

## The Multiplier Theorem

**Thm:** Let $m : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ satisfy, for any multi-index, $\gamma$, of length $|\gamma| \le d+2$

$$|\partial^\gamma m(\xi)| \le B |\xi|^{-|\gamma|}$$

Then for any $1 < p < \infty$, $\exists\, C = C(d,p)$ s.t.

$$\|(m\hat{f})^\vee\|_p \le C B \|f\|_p \qquad \forall f \in \mathcal{S}(\mathbb{R}^d)$$

## pf:

For $j \in \mathbb{Z}$, let

$$m_j(\xi) = \psi(2^{-j}\xi)\, m(\xi)$$

and $$K_j = \check{m}_j$$

Fix $N \in \mathbb{Z}_+$ and define

$$K = \sum_{j=-N}^{N} K_j$$

**Claim:** i.) $|K(x)| \le C B |x|^{-d}$

ii.) $|\nabla K(x)| \le C B |x|^{-(d+1)}$

pf of claim

$$\| D^\gamma m_j \|_\infty \lesssim \| 2^{-|\gamma| j} (D^\gamma \psi)\, m \|_\infty + \| D^\gamma m \|_{L^\infty(|\xi| \geq 2^{j+2})}.$$

$$\lesssim 2^{-j|\gamma|}$$

and

$$\| D^\gamma m_j \|_{L^2} \lesssim 2^{jd} \cdot 2^{-j|\gamma|}$$

Similarly, for $i = 1, \dots, d$

$$\| D^\gamma (\xi_i m_j) \|_{L^1} \lesssim 2^{-j|\gamma|} 2^{jd} 2^{-j}$$

Thus

$$\| x^\gamma \check{m}_j \|_\infty \lesssim 2^{jd} 2^{-|\gamma| j}$$

and

$$\| x^\gamma D \check{m}_j \|_\infty \lesssim 2^{jd} 2^{-|\gamma| j} 2^{+j}$$

$$\Rightarrow \quad | \check{m}_j (x) | \lesssim 2^{j(d-k)} |x|^{-k}$$

And

$$| D \check{m}_j (x) | \lesssim 2^{j(d-k)} 2^{+j} |x|^{-k}$$

Thus, for $x \in \mathbb{R}^d \setminus \{0\}$

$$|K(x)| \leq \sum_{j=-N}^{N} | \check{m}_j (x) | = \sum_{|x| \leq 2^{-j}} | \check{m}_j (x) | + \sum_{|x| > 2^{-j}} | \check{m}_j (x) |$$

$$\lesssim \sum_{|x| \leq 2^{-j}} \| m_j \|_{L^1} + \sum_{|x| > 2^{-j}} 2^{j(d-(d+2))} |x|^{-(d+2)}$$

$$\lesssim \sum_{2^j \leq |x|^{-1}} 2^{jd} + \sum_{2^j > |x|^{-1}} 2^{-2j} |x|^{-(d+2)}$$

$$\lesssim |x|^{-d}$$

Similarly

$$|\nabla K(x)| \lesssim |x|^{-(d+1)}$$

Thus, $K$ is a Calderon-Zygmund Kernel and the estimates

- $|K(x)| \leq C|x|^{-d}$
- $|\nabla K(x)| \leq C |x|^{-(d+1)}$

do not depend on $N$.

Thus

$$\|(m\hat{f})^{\vee}\|_p \leq \sup_N \|K * f\|_p \leq C \|f\|_p .$$

# The Littlewood-Paley Square Function

Much of what we've studied and will continue to study revolves around the idea of discretization of functions. The act of discretizing a function in physical space is quite natural. Euclidean space is locally, countably compact, so one can split the space into countably many intervals or cubes. Restricting a function to these cubes immediately provides some independence

$$\|f\|_{L^p}^p = \sum_I \|\chi_I f\|_p^p$$

This doesn't work as nicely.

We need to use a delicate notion of frequency independence:

Let $\psi_j(x) = \psi(2^{-j}x)$

where $\sum \psi(2^{-j}x) = 1 \quad \forall x \neq 0$

Define the $j$th Littlewood-Paley projector by

$$P_j f := (\psi_j \hat{f})^\vee$$

By Plancherel

$$\sum_j \|P_j f\|_2^2 \sim \|f\|_2^2$$

Define the Littlewood-Paley Square Function by

$$Sf(x) = \left( \sum_{j \in \mathbb{Z}} |P_j f|^2 \right)^{1/2}$$

Now we have the following theorem

<u>Thm:</u> (Littlewood-Paley)

For any $1<p<\infty$ $\exists C = C(p,d)$ s.t.

$$C^{-1}\|f\|_p \le \|Sf\|_p \le C\|f\|_p \quad \text{for } f\in S(\mathbb{R}^d)$$

---

<u>Pf:</u>

Let $\{r_j\}$ be the sequence of Rademacher functions, let $N\in\mathbb{Z}$ and

$$m(\xi) := \sum_{j=-N}^{N} r_j(t)\,\psi_j(\xi)$$

Then

$$|D^\gamma m(\xi)| \le \sum_{j=-N}^{N} |D^\gamma \psi_j(\xi)|$$

$$\lesssim \sum_{j=-N}^{N} |\xi|^{-|\gamma|}\,|(D^\gamma\psi)(2^{-j}\xi)|$$

$$\le C|\xi|^{-|\gamma|}$$

Now, Khinchine's inequality provides

$$\int |Sf(x)|^p$$

$$\le \limsup_{N\to\infty} \int \Big| \sum_{j=-N}^{N} |P_j f(x)|^2 \Big|^{p/2}$$

Khinchine $\le \limsup_{N\to\infty} \mathbb{E} \int \Big| \sum_{j=-N}^{N} r_j P_j f \Big|^p$

Multiplier Thm $\lesssim C \|f\|_p^p$

For the lower bound, consider a L-P projector $\tilde{P}_j$ s.t. $\tilde{P}_j P_j = P_j$

Then

$$|\langle f, g\rangle| = \Big| \sum_j \langle P_j f, \tilde{P}_j g \rangle \Big|$$

Cauchy-Schwarz $\le \int_{\mathbb{R}^d} \Big(\sum_j |P_j f(x)|^2\Big)^{1/2} \Big(\sum_k |\tilde{P}_k g(x)|^2\Big)^{1/2}$

Hölder $\le \|Sf\|_p \|\tilde{S} g\|_{p'} \lesssim \|Sf\|_p \|g\|_{p'}$ □