Cor (Khinchine's Inequality)

For any $p \in [1,\infty)$ there exists a constant $C = C(p)$ s.t.

$$C^{-1}\left(\sum_{j=1}^{N}|a_j|^2\right)^{p/2} \le \mathbb{E}\left|\sum_{j=1}^{N} a_j r_j\right|^p$$

$$\le C\left(\sum_{j=1}^{N}|a_j|^2\right)^{p/2}.$$

for any $N \in \mathbb{Z}_+$ and $\{a_j\}_{j=1}^{N} \subset \mathbb{R}$

pf:

① $\mathbb{E}\left|\sum a_j r_j\right|^p \le C\left(\sum_{j=1}^{N}|a_j|^2\right)^{p/2}$

Let $\sigma = \left(\sum_{j=1}^{N}|a_j|^2\right)^{1/2}$

and $S_N(t) = \sum_{j=1}^{N} a_j r_j(t)$

$$\sigma^p\,\mathbb{E}\left|\sum a_j r_j\right|^p = \int_0^\infty P(\{|S_N| > \lambda\sigma\})\, p\lambda^{p-1} d\lambda$$

$$\le \int_0^\infty 4e^{-\lambda^2/4}\, p\lambda^{p-1} = C(p) < \infty$$

$$\Rightarrow \quad \mathbb{E}|S_N|^p \le C(p)\,\sigma^p$$

$$② \quad C^{-1}\Big(\sum_{j=1}^{N}|a_j|^2\Big)^{p/2} \le \mathbb{E}\Big|\sum a_j r_j\Big|^p$$

Observe that

$$\sum_{j=1}^{N}|a_j|^2 = \mathbb{E}|S_N|^2$$

and for $\frac{1}{p}+\frac{1}{q}=1$, $p\in(1,\infty)$

$$\sum_{j=1}^{N}|a_j|^2 = \mathbb{E}|S_N|^2 \le \left(\mathbb{E}|S_N|^p\right)^{1/p}\left(\mathbb{E}|S_N|^q\right)^{1/q}$$

by part (1) $\le \left(\mathbb{E}|S_N|^p\right)^{1/p}\,\sigma\cdot C(q)^{1/q}$

$$\Rightarrow \quad C(q)^{-p/q}\Big(\sum_{j=1}^{N}|a_j|^2\Big)^{p/2} \le \mathbb{E}|S_N|^p$$

For $p=1$

$$\sum_{j=1}^{N}|a_j|^2 = \mathbb{E}|S_N|^2 = \mathbb{E}\,|S_N|^{1/2}\,|S_N|^{3/2}$$

$$\le \left(\mathbb{E}|S_N|\right)^{1/2}\left(\mathbb{E}|S_N|^3\right)^{1/2}$$

$$\leq \left(\mathbb{E}|S_N|\right)^{1/2} \sigma^{3/2} \cdot C(3)^{1/2}$$

$$\Rightarrow \quad C(3)^{-1/2} \sigma^{1/2} \leq \left(\mathbb{E}|S_N|\right)^{1/2}$$

$$\Rightarrow \quad C(3) \sigma \leq \mathbb{E}|S_N|$$

$\square$.

Littlewood-Paley Theory

Consider a bounded measurable function $m : \mathbb{R} \to \mathbb{C}$, and an associated multiplier operator:

$$T_m f(x) = \int e(-x\xi)\, m(\xi)\, \hat{f}(\xi)\, d\xi$$

Of course,

$$\|T_m\|_{L^2 \to L^2} \leq \|m\|_\infty .$$

and

$$T_m f(x) = (K_m * f)(x) \quad \text{where}$$

$$K_m = \check{m} .$$

So Fourier multiplier operators

are convolution operators.

We will use Fourier multiplier operators to build discrete wavelet theory.

Lemma (Geometric Partition of Unity)

$\exists\ \psi \in C^\infty(\mathbb{R})$ with the property that $\operatorname{supp}(\psi) \subset \mathbb{R} \setminus \{0\}$ is compact and

$$\sum_{j=-\infty}^{\infty} \psi(2^{-j}x) = 1 \quad \forall x \neq 0.$$

For any given $x \neq 0$, at most two terms in this sum are nonzero. Moreover, ψ can be chosen to be a radial nonnegative function.

Pf: Let $\varphi \in C^\infty_c(\mathbb{R})$ satisfy

$$\varphi(x) = \begin{cases} 0 & |x| > 2 \\ 1 & |x| \le 1. \end{cases}$$

Now define $\psi(x) = \varphi(x) - \varphi(2x)$

Then
$$\sum_{j=-M}^{M} \psi(2^{-j}x) = \varphi(2^{-M}x) - \varphi(2^{M}x)$$
$$= \begin{cases} 0, & |x| < 2^{-M},\ |x| > 2^{M+1} \\ 1, & 2^{-M+1} < |x| < 2^{M}. \end{cases}$$

$\square$.

The Multiplier Theorem

<u>Thm:</u> Let $m : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ satisfy, for any multi-index, γ, of length $|\gamma| \le d+2$

$$|\partial^\gamma m(\xi)| \le B|\xi|^{-|\gamma|}$$

Then for any $1 < p < \infty$, $\exists\, C = C(d,p)$ s.t.

$$\|(m\hat{f})^\vee\|_p \le C B \|f\|_p \qquad \forall f \in \mathcal{S}(\mathbb{R}^d)$$