A sequence of independent, identically distributed random variables (i.i.d.) is a sequence $\{X_j\}_{j=1}^{\infty}$ s.t. $\exists\, F : \mathbb{R} \to [0,1]$, non-increasing, right-continuous and $\lim_{\lambda \to -\infty} F(\lambda) = 1$ and $\lim_{\lambda \to +\infty} F(\lambda) = 0$ s.t.

$$\mathbb{P}(\{X_j > \lambda\}) = F(\lambda) \quad \text{for all } j \text{ and}$$

$$\begin{aligned} F_N(\lambda_1, \ldots, \lambda_N) &= \mathbb{P}(\{X_1 > \lambda_1, X_2 > \lambda_2, \ldots, X_N > \lambda_N\}) \\ &= \prod_{j=1}^{N} \mathbb{P}(\{X_j > \lambda_j\}) \\ &= \prod_{j=1}^{N} F(\lambda_j). \end{aligned}$$

An example of an i.i.d. is the Rademacher sequence.

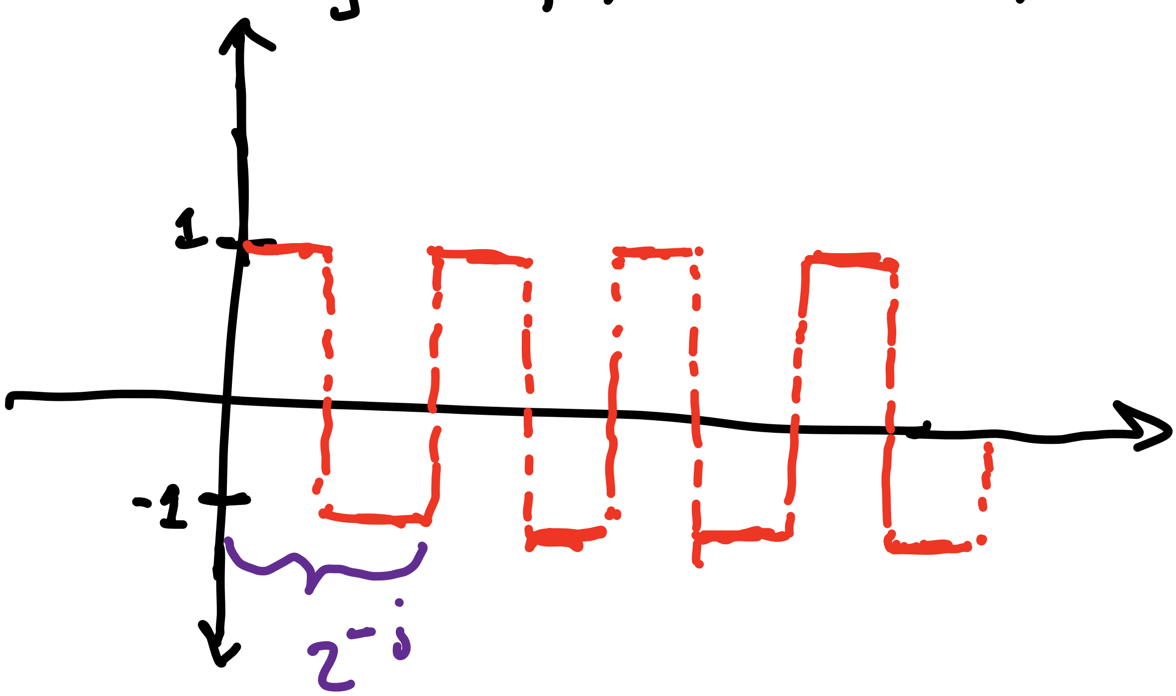
Rademacher Sequence

We encode a sequence of independent coin flips by the Rademacher functions on the measure space $([0,1], \mathcal{L}) = (\Omega, \mathbb{P})$.

Define

$$r_j(t) := \operatorname{sign}(\sin(2\pi\, 2^j t))$$

$$j = \{0, 1, \dots\}, \quad t \in [0,1].$$



Lemma:

Let $\{r_j\}$ be the Rademacher sequence..

For $N \in \mathbb{Z}_+$ and $\{a_j\}_{j=1}^{N} \subset \mathbb{C}$

on has

$$\mathbb{P}\left(\left\{ \left|\sum_{j=1}^{N} r_j a_j\right| > \lambda \left(\sum_{j=1}^{N} |a_j|^2\right)^{1/2} \right\}\right) \leq 4^{-\lambda^2/4}$$

for all $\lambda > 0$.

Pf:

$$\mathbb{P}\left(\left\{ \left|\sum_{j=1}^{N} r_j a_j\right| > \lambda \left(\sum_{j=1}^{N} |a_j|^2\right)^{1/2} \right\}\right)$$

$$\leq \mathbb{P}\left(\left\{ \left|\sum_{j=1}^{N} r_j \operatorname{Re}(a_j)\right| > \frac{\lambda}{2} \left(\sum_{j=1}^{N} |a_j|^2\right)^{1/2} \right\}\right)$$

$$+ \mathbb{P}\left(\left\{ \left|\sum_{j=1}^{N} r_j \operatorname{Im}(a_j)\right| > \frac{\lambda}{2} \left(\sum_{j=1}^{N} |a_j|^2\right)^{1/2} \right\}\right)$$

For $t > 0$ and $S_N = \sum_{j=1}^{N} r_j \operatorname{Re}(a_j)$

we have

$$\mathbb{E}\, e^{tS_N} = \prod_{j=1}^{N} \mathbb{E}\left(e^{t r_j \operatorname{Re} a_j}\right)$$

$$= \prod_{j=1}^{N} \cosh(t \operatorname{Re}(a_j))$$

Now note that $\cosh(x) \le e^{x^2/2}$ $\forall x \in \mathbb{R}$

$\Rightarrow$

$$\mathbb{E}\, e^{tS_N} \le \prod_{j=1}^{N} \exp\left(t^2 (\operatorname{Re}(a_j))^2 / 2\right)$$

$$\le \exp\left(t^2 \frac{\sum |a_j|^2}{2}\right)$$

Then, if $\sigma = \left(\sum |a_j|^2\right)^{1/2}$

$$\mathbb{P}(\{S_N > \tfrac{\lambda}{2}\sigma\})$$

$$= \mathbb{P}\left(\{\exp(tS_N) \ge \exp(t\tfrac{\lambda\sigma}{2})\}\right)$$

$$\le \frac{\mathbb{E}[\exp(tS_N)]}{\exp(t\frac{\lambda}{2}\sigma)} \le \frac{\exp\left(\frac{t^2}{2} \sum |a_j|^2\right)}{\exp(t\lambda\sigma/2)}.$$

$$= \frac{\exp\left(\frac{t^2}{2}\sigma^2\right)}{\exp(t\lambda\sigma/2)}$$

Let $t = \frac{\lambda}{2\sigma}$, then

$$\mathbb{P}(\{ S_N > \tfrac{\lambda}{2}\sigma \}) \le \exp\left(\frac{\lambda^2}{8} - \frac{\lambda^2}{4} \right)$$

$$= \exp(-\lambda^2/4)$$

Similarly

$$\mathbb{P}(\{ S_N < -\tfrac{\lambda}{2}\sigma \}) \le \exp(-\lambda^2/4).$$

$$\Rightarrow \mathbb{P}(\{ |S_N| > \tfrac{\lambda}{2}\sigma \}) \le 2\exp(-\lambda^2/4)$$

Using the same argument for the imaginary part yields

$$\mathbb{P}\{\{ |\Sigma r_j a_j| > \lambda\sigma \}) \le 4\exp(-\lambda^2/4).$$

$\square$

The sub-Gaussian estimate implies all moments are equivalent