L^2 Sobolev Spaces

For $f \in \mathcal{S}(\mathbb{R})$, define

$$\|f\|_{H^s} := \|\langle\xi\rangle^s \hat{f}\|_{L^2} \quad \text{where}$$

$$\langle\xi\rangle = (1+|\xi|^2)^{1/2}$$

and

$$\|f\|_{\dot{H}^s} := \| \, |\xi|^s \hat{f} \, \|_{L^2}$$

Then for $s \in \mathbb{R}$,

$H^s(\mathbb{R}) = \overline{\mathcal{S}(\mathbb{R})}$ under topology defined by $\|\cdot\|_{H^s}$

and for $s > -d/2 = -1$

$\dot{H}^s(\mathbb{R}) = \overline{\mathcal{S}(\mathbb{R})}$ under topology defined by $\|\cdot\|_{\dot{H}^s}$.

Lemma (L^2 Sobolev Embedding)

For any $f \in H^s(\mathbb{R})$

$$\|f\|_p \leq C_s \|f\|_{H^s(\mathbb{R})}$$

for all $2 \leq p \leq \infty$

provided that $s > d/2 = 1$.

In fact, $H^s(\mathbb{R}) \hookrightarrow C(\mathbb{R}) \cap L^p(\mathbb{R})$.

Pf: Cauchy-Schwarz

$$\|f\|_\infty \le \|\hat{f}\|_1 \le \|\langle\xi\rangle^{-s}\|_2 \|\langle\xi\rangle^s \hat{f}(\xi)\|_2.$$

and

$$\|f\|_2 \le \|\langle\xi\rangle^{-s}\|_\infty \|\langle\xi\rangle^s \hat{f}(\xi)\|_2.$$

$$\Rightarrow \quad \|f\|_p \le \|f\|_2^\alpha \|f\|_\infty^{1-\alpha} \le C_s \|f\|_{H^s}. \quad \square$$

Corollary:

Let $f \in \mathcal{S}(\mathbb{R})$, $\operatorname{supp}(\hat{f}) \subset [-N, N]$.

$$\|f\|_\infty \lesssim N^{1/2+} \|f\|_2.$$

Basic Probability.

Notation

A probability space $(\Omega, \Sigma, \mathbb{P})$ is a measure space with a positive measure, $\mathbb{P}$, s.t. $\mathbb{P}(\Omega) = 1$.

$A, B \in \Sigma$ are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Finitely many σ-algebras, Σ_j, are independent if for any $A_j \in \Sigma_j$

$$\mathbb{P}\left(\bigcap_j A_j\right) = \prod_j \mathbb{P}(A_j)$$

and finitely many random variables, X_j, are independent if the σ-algebras, $\{X_j^{-1}(\mathcal{B})\}_j$, are independent.

Lemma (Borel-Cantelli)

Let $\{A_j\}_{j=1}^{\infty} \subset \Sigma$, then

$$\sum_{j=1}^{\infty} \mathbb{P}(A_j) < \infty \Rightarrow \mathbb{P}(A_j \text{ occurs infinitely often}) = 0$$

Now assume, in addition, that the A_j's are independent. Then

$$\sum_{j=1}^{\infty} \mathbb{P}(A_j) \infty \Rightarrow \mathbb{P}(A_j \text{ occurs infinitely often}) = 1.$$

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pf: The first part follows from monotone convergence theorem:

monotone convergence thm.

$$\sum_{j=1}^{\infty} \mathbb{P}(A_j) = \sum_{j=1}^{\infty} \int \chi_{A_j} \, d\mathbb{P} = \int \sum_{j=1}^{\infty} \chi_{A_j} \, d\mathbb{P} < \infty.$$

$$\Rightarrow \mathbb{P}(\{\sum \chi_{A_j} = \infty\}) = 0$$

$$\Rightarrow \mathbb{P}(A_j \text{ occurs infinitely often}) = 0.$$

For the second part,

$$\mathbb{P}\left(\bigcap_{j=1}^{N} A_j^c\right) = \prod_{j=1}^{N} \left(1 - \mathbb{P}(A_j)\right)$$

$$\leq \prod_{j=1}^{N} \exp\left(-\mathbb{P}(A_j)\right)$$

$$= \exp\left(-\sum_{j=1}^{N} \mathbb{P}(A_j)\right) \xrightarrow{N\to\infty} 0.$$

Thus,

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} A_j\right) = 1 \quad \text{for all } k.$$

$\square$.