Schur's Lemma:

To generalize the convolution-type singular integral operators, consider a product measure space

$$(X \times Y, \mu \otimes \nu) \text{ and a kernel}$$

$$K : X \times Y \to \mathbb{C}$$

We can now ask how to bound operators of the form

$$(Tf)(x) := \int_Y K(x,y) f(y) \, \nu(dy)$$

for $f \in L^p(d\nu)$.

Hölder's inequality allows for

$$\|Tf\|_{L^p(d\mu)} = \left(\int \left| \int K(x,y) f(y) \, \nu(dy) \right|^p \mu(dx) \right)^{1/p}$$

$$\leq \left(\int \|K(x,\cdot)\|^p_{L^q(d\nu)} \cdot \|f\|^p_{L^p(d\nu)} \, \mu(dx) \right)^{1/p}$$

$$\leq \|K\|_{L^p(d\mu;\, L^q(d\nu))} \|f\|_{L^p(d\nu)}$$

where $\frac{1}{p} + \frac{1}{q} = 1.$

when $p = 2 = q$ and

$$\|T\|_{L^2(d\nu) \to L^2(d\mu)} \leq \|K\|_{L^2(d\mu \otimes d\nu)}$$

and $\|K\|_{L^2}$ is called the Hilbert-Schmidt Norm

An easier way to control the operator norm is Schur's Lemma:

Lemma:

Let $Tf(x) := \int_Y K(x,y) f(y)\, \nu(dy)$ where K is measurable. Then

i.) $\|T\|_{L^1(d\nu) \to L^1(d\mu)} \leq \sup_{y \in Y} \int_X |K(x,y)|\, \mu(dx) =: A$

ii.) $\|T\|_{L^\infty(d\nu) \to L^\infty(d\mu)} \leq \sup_{x \in X} \int_Y |K(x,y)|\, \nu(dy) =: B.$

iii.) $\|T\|_{L^p(d\nu) \to L^p(d\mu)} \leq A^{1/p} B^{1/p'}$ where $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1.$

iv.) $\|T\|_{L^1(d\nu) \to L^\infty(d\mu)} \leq \|K\|_{L^\infty(X \times Y)}.$

The proof is Hölder and interpolation.

Now we know enough to answer the question of whether $\operatorname{supp} f \subset E$ and $\operatorname{supp}(\hat{f}) \subset F$ is possible.

Suppose $|E|, |F| < \infty$. Suppose $\exists f \in L^2(\mathbb{R}^d)$ such that

$$\operatorname{supp}(f) \subset E$$

and

$$\operatorname{supp}(\hat{f}) \subset F.$$

If that were the case, then

$$\chi_E (\chi_F \hat{f})^\vee = f$$

Let

$$Tf = \chi_E (\chi_F \hat{f})^\vee$$

Then

$$Tf = \chi_E \cdot (\check{\chi}_F * f)$$

and

$$Tf(x) = \int_{\mathbb{R}} \chi_E(x)\, \check{\chi}_F(x-y) f(y)\, dy$$

and if

$$K(x,y) = \chi_E(x)\, \check{\chi}_F(x-y)$$

$$Tf(x) = \int_{\mathbb{R}} K(x,y) f(y)\, dy$$

and

$$\|T\|^2_{L^2 \to L^2} \leq \int_{\mathbb{R}\times\mathbb{R}} |K(x,y)|^2 \,dx\,dy$$

by Plancherel $= |E|\,|F|$

$\Rightarrow \quad \|T\| \leq \left(|E|\,|F|\right)^{1/2}.$

but if $Tf = f$ for some f,

$$\|T\| \geq 1.$$

$\Rightarrow \quad 1 \leq \left(|E| \cdot |F|\right)^{1/2}.$

$\Rightarrow$ If $|E| \cdot |F| < 1$, then there does not exist $f \in L^2(\mathbb{R})$ such that

$\operatorname{supp}(f) \subset E$ and $\operatorname{supp}(\hat{f}) \subset F$.

This actually holds for any

E, F with $|E| < \infty$ and $|F| < \infty$

<u>Thm</u>: Let E and F be sets of finite measure in $\mathbb{R}^d$. Then

$$\|f\|_{L^2(\mathbb{R}^d)} \le C\left(\|f\|_{L^2(E^c)} + \|\hat{f}\|_{L^2(F^c)}\right)$$

for some $C = C(E,F,d)$.

<u>pf</u>: Note that $Tf = \chi_E(\chi_F \hat{f})^\vee$ is a compact operator. Therefore,

$\{\text{eigenvalues of } T\} \setminus \{0\} = \text{spectrum of } T \setminus \{0\}$.

and $\sigma(T) \subset \{z \in \mathbb{C} \mid |z| \le \|T\|\}$

For $\lambda \le \|T\| \le 1$, let $\{f_j\}_{j=1}^m$ be an orthonormal sequence of eigenfunctions for T such that

$$\lambda \le |\langle Tf_j, f_j \rangle|$$
$$= \left| \int_{\mathbb{R}^{2d}} K(x,y) f_j(x) f_j(y) \right|$$

Thus

$$m\lambda^2 \le \sum_{j=1}^{m} \left| \int_{\mathbb{R}^{2d}} K(x,y) f_j(x) f_j(y) \right|^2$$

$$\le \|K\|^2_{L^2(\mathbb{R}^{2d})}.$$

Thus

$$\dim\left(\{ f \in L^2(\mathbb{R}^d) \,|\, |Tf| \ge \lambda |f| \} \right) \le \lambda^{-2} \|K\|^2_{L^2}$$

However if $\lambda = 1$, $f_0 \in \{ f \in L^2(\mathbb{R}^d) \,|\, |Tf| \ge \lambda |f| \}$

and let $\{x_k\}_{k=1}^{\infty} \subset B(0,1)$

s.t. $x_{k+1} + \mathrm{supp}(f_0) \not\subset \bigcup_{\ell=1}^{k} x_\ell + \mathrm{supp}(f_0)$.

and $\left| \chi_E f(x - x_{k+1}) \right| \ge \lambda | f(x - x_k) |$

Thus, if $f_k = f(\cdot - x_k)$, then $f_k \in \{ |Tf| \ge \lambda |f| \}$

Since $f_k = f(\cdot - x_{k+1})$ are linearly independent,

$$\infty = \dim \{ f \in L^2(\mathbb{R}^d) \mid |Tf| \geq \lambda |f| \}$$

$$\leq \lambda^{-2} \|K\|^2 < \infty$$

which gives a contradiction.

Thus

$$\| Tf \|_2 \leq \rho \|f\|_2 \quad \text{for some } \rho < 1.$$

$\Rightarrow$ If $\operatorname{supp}(\hat{f}) \subset F$, then

$$\| Tf \|_2^2 = \| \chi_E (\chi_F \hat{f})^\vee \|_2^2$$

$$= \| \chi_E f \|_2^2 = \|f\|_2^2 - \| \chi_{E^c} f \|_2^2$$

$\Rightarrow$

$$\| f \|_2^2 = \| \chi_{E^c} f \|_2^2 + \| Tf \|_2^2$$

$$\leq \| \chi_{E^c} f \|_2^2 + \rho^2 \| f \|_2^2$$

$\Rightarrow$

$$\| f \|_2 \leq (1-\rho^2)^{-1/2} \| \chi_{E^c} f \|_2.$$

Therefore,

$$\|f\|_2 \le \| (\chi_F \hat{f})^\vee \|_2 + \| (\chi_{F^c} \hat{f})^\vee \|_2$$

$$\le (1-\rho^2)^{-1/2} \| \chi_{E^c} (\chi_F \hat{f})^\vee \|_2 + \| (\chi_{F^c} \hat{f})^\vee \|_2$$

$$\le (1-\rho^2)^{-1/2} \| \chi_{E^c} f \|_2 + (1-\rho^2)^{-1/2} \| \chi_{E^c} (\chi_{F^c} \hat{f})^\vee \|_2$$

$$+ \| \hat{f} \|_{L^2(F^c)}$$

$$\le (1-\rho^2)^{-1/2} \| f \|_{L^2(E^c)} + \left[(1-\rho^2)^{-1/2} + 1 \right] \| \hat{f} \|_{L^2(F^c)}.$$

$\square$.