A Note on Differentiation

Of course, we have the classical derivative for $f \in C^\infty(\mathbb{R})$, which is local,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

and we have the distributional notion of differentiation for $f \in L^1_{loc}$, defined by the identity

$$\int f(x)\,\phi'(x)\,dx = -\int \partial f(x)\,\phi(x)\,dx$$

for all $\phi \in C_c^\infty(\mathbb{R})$. For $k \in \mathbb{Z}_+$, ∂^k is the natural higher derivative.

Now for $f \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} f'(x) = \partial f(x) &= \int \hat{f}(\xi) \frac{d}{dx} e(x\xi)\,d\xi \\ &= \int 2\pi i \xi \hat{f}(\xi) e(x\xi)\,\xi \\ &= (2\pi i \xi \hat{f})^\vee(x). \end{aligned}$$

So differentiation in physical space is equivalent to multiplying by $2\pi i\xi$ is frequency space (and vice versa).

This is a nonlocal version of differentiation because in order to compute the Fourier transform, we need to use information for f from its entire support. The classical derivative only needs local information.

We also have the normalized derivative operator:

$$Df(x) := \left(\frac{1}{2\pi i}\right)\partial f(x).$$

$$\Rightarrow \quad \widehat{Df}(\xi) = \xi\hat{f}(\xi).$$

Lemma: Let $f \in L^2$ and $\operatorname{supp}(\hat{f}) \subset B(0,R)$ then

$$\|D^k f\|_{L^2} \leq R^k \|f\|_{L^2}.$$

Heisenberg's Uncertainty Principle

Prop: Let $f \in \mathcal{S}(\mathbb{R})$, then

$$\|f\|_2^2 \leq 4\pi \,\|(x-x_0)f\|_2 \,\|(\xi-\xi_0)\hat{f}\|_2.$$

for all $x_0, \xi_0 \in \mathbb{R}$. This inequality is sharp, the extremizers being

$$f(x) = C e(\xi_0 x)\, e^{-\pi\delta(x-x_0)^2}$$

where $C \in \mathbb{C}$, $\delta > 0$.

Pf WLOG, let $x_0 = \xi_0 = 0$.

Let $$D = \frac{1}{2\pi i}\frac{d}{dx}.$$

and $$(Xf)(x) := x f(x).$$

Then

$$[D, X] = DX - XD = \frac{1}{2\pi i}$$

Thus, for $f \in \mathcal{S}(\mathbb{R})$

$$\|f\|_2^2 = 2\pi i \langle [D,x] f, f \rangle$$

$$= 2\pi i \left(\langle xf, Df \rangle - \langle Df, xf \rangle \right)$$

$$= 4\pi \, \mathrm{Im}\left(\langle Df, xf \rangle \right)$$

$$\leq 4\pi \|Df\|_2 \|xf\|_2 .$$

Sharpness: Exercise.

The Amrein-Berthier theorem

Q: If $E, F \subset \mathbb{R}$ are of finite Lebesgue measure, can there be a nonzero $f \in L^2(\mathbb{R})$ with $\mathrm{supp}(f) \subset E$ and $\mathrm{supp}(\hat{f}) \subset F$?

In order to answer this question, we need to further develop singular integral theory.

Cotlar's Lemma

First, a very general lemma

Lem: Let $\{T_j\}_{j=1}^N$ be finitely many operators on some Hilbert space, $\mathcal{H}$, such that for some function, $\gamma : \mathbb{Z} \to \mathbb{R}_+$, one has

$$\| T_j^* T_k \| \leq \gamma^2(j-k) \quad , \quad \| T_j T_k^* \| \leq \gamma^2(j-k)$$

for any $1 \leq j, k \leq N$. Let

$$\sum_{l \in \mathbb{Z}} \gamma(l) =: A < \infty$$

Then

$$\| \sum_{j=1}^N T_j \| \leq A \; .$$

Pf; Let $T = \sum_{j=1}^N T_j$, then for $n \in \mathbb{Z}_+$

$$(T^* T)^n = \sum T_{j_1} T^*_{k_1} T_{j_2} T^*_{k_2} \cdots T_{j_n} T^*_{k_n}$$

Observe that

$$\|T_{j_1}T_{k_1}^* \cdots T_{j_n}T_{k_n}^*\|$$

$$= \|T_{j_1}(T_{k_1}^*T_{j_2})(\cdots)(T_{k_{n-1}}^*T_{j_n})T_{k_n}^*\|$$

$$\leq \|T_{j_2}\|\,\|T_{k_n}^*\| \prod_{i=1}^{n-1} \|T_{k_i}^*T_{j_{i+1}}\|$$

and

$$\|T_{j_1}T_{k_1}^* \cdots T_{j_n}T_{k_n}^*\| \leq \prod_{i=1}^{n} \|T_{j_i}T_{k_i}^*\|$$

$\Rightarrow$

$$\|T_{j_1}T_{k_2}^* \cdots T_{j_n}T_{k_n}^*\| \leq (\|T_{j_1}\|\,\|T_{k_n}^*\|)^{1/2} \prod_{i=1}^{n-1} \|T_{k_i}^*T_{j_{i+1}}\|^{1/2} \cdot \prod_{i=1}^{n} \|T_{j_i}T_{k_i}^*\|^{1/2}.$$

Then, write $\sup_{1\le j\le N}\|T_j\| =: B \le A$,

$\|(T^*T)^n\|$

$$\le \sum \|T_{j_1}\|^{1/2}\|T_{j_1}^*T_{k_1}\|^{1/2} \cdots \|T_{k_{n-1}}T_{j_n}^*\|^{1/2} \cdot \|T_{j_n}^*T_{k_n}\|^{1/2} \cdot \|T_{k_n}\|^{1/2}$$

$$\le \sum \sqrt{B}\, \gamma(j_1-k_1)\gamma(k_1-j_2)\gamma(j_2-k_2)\cdots\gamma(j_n-k_n)\sqrt{B}.$$

$$\le NB\Big(\sum_{\ell\in\mathbb{Z}}\gamma(\ell)\Big)^{2n} \le NB(A^{2n})$$

Since T^*T is self-adjoint,

$$\|T^*T\|^n = \|(T^*T)^n\| \le NBA^{2n}.$$

$\Rightarrow$

$$\|T\| \le (NB)^{1/2n}\cdot A.$$

and as $n\to\infty$

$$\|T\| \le A \qquad \square.$$