We know develop our understanding of wave packets by starting with essential propositions and theorems in Fourier Analysis.

Bernstein's Inequality

Prop: Let $f \in S(\mathbb{R})$, $\operatorname{supp} \hat{f} \subset [-N, N]$.

Then
$$\|f'\|_{L^p} \lesssim N \|f\|_{L^p}$$

for $p \in [1, \infty]$.

<u>Pf:</u> Let $\psi \in S(\mathbb{R})$ satisfy

- $0 \leq \psi(\xi) \leq 1$ for all $\xi \in \mathbb{R}$
- $\psi = 1$ on $[-1, 1]$
- $\psi = 0$ on $[-2, 2]^c$

For $N > 0$, define

$$\psi_N(\xi) = \psi(N^{-1}\xi)$$

and

$$\phi_N(x) := \int_{\mathbb{R}} e(x\xi)\, \psi(N^{-1}\xi)\, d\xi.$$

$$= N \int_{\mathbb{R}} e(Nx\xi)\, \psi(\xi)\, d\xi.$$

$$= N\, \phi_1(Nx).$$

Then, since $\widehat{\phi_N}(\xi) = 1$ on $[-N, N]$,

$$f = f * \phi_N \quad \text{and} \quad \frac{d}{dx}\phi_N = N\left(N\, \phi_1'(Nx)\right).$$

$$\| N\, \phi_1'(Nx) \|_1 = \| \phi_1' \|_{L^1} \lesssim 1.$$

$\Rightarrow$

$$\| f' \|_p = \| \phi_N' * f \|_p \leq \| \phi_N' \|_1 \, \| f \|_p$$

$$\lesssim N \, \| f \|_p \quad \square.$$