Our goal is to study Fourier Analysis through the lens of searching for the answer to the question of convergence of Fourier series.

We will focus on two regimes.

1. $f : \mathbb{R} \to \mathbb{C}$, $f \in L^1(\mathbb{R})$

$$\hat{f}(\xi) := \int_{\mathbb{R}} e^{-i 2\pi x \xi} f(x)\, dx$$

$\hat{f} : \mathbb{R} \to \mathbb{C}$

2. $f : \mathbb{T} \to \mathbb{C}$, $f \in L^1(\mathbb{T})$

$$\hat{f}(n) := \int_{\mathbb{T}} e^{-2\pi i n x} f(x)\, dx.$$

$\hat{f} : \mathbb{Z} \to \mathbb{C}.$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}.$

We need $f \in L^1$ simply to make sense of the Fourier transform.

We now need to define the central object of this course:

Partial Sums Operator

Let $e(x) := e^{i2\pi x}$

Suppose

$$g(x) = \sum_{k=-M}^{M} a_k \, e(kx) \qquad x \in \mathbb{T}$$

Since

$$\int_{\mathbb{T}} e(-nx)\, e(mx) = \delta_0(n-m),$$

$$\hat{g}(k) = a_k$$

Therefore,

$$g(x) = \sum_{k=-M}^{M} \hat{g}(k)\, e(kx) = \sum_{k=-\infty}^{\infty} \hat{g}(k)\, e(kx).$$

The question is to what extent

$$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k)\, e(kx) \qquad \text{for } f \in L^1(\mathbb{T})$$

A natural question is then, if

$$S_N f(x) := \sum_{k=-N}^{N} \hat{f}(k)\, e(kx).$$

then does

$$\lim_{N \to \infty} S_N f = f \quad \text{hold?}$$

And in what sense?

Some Basics.

Function Spaces.

For $1 \le p < \infty$, for $X = \mathbb{T}$ or $X = \mathbb{R}$.

$$L^p(X) := \left\{ f : X \to \mathbb{C} \;\middle|\; \int_X |f(x)|^p \, dx < \infty \right\}.$$

$$\|f\|_{L^p(X)} = \|f\|_p := \left(\int_X |f|^p \, dx \right)^{1/p}$$

Weak-L^p : $\sup_{\lambda > 0} \lambda^p |\{ |f| > \lambda \}| < \infty$.

$C^\infty(X) :=$ { infinitely differentiable functions on X }

$C(X) =$ { continuous functions on X }.

$C_c(X) =$ { continuous functions w/ compact support on X }.

For $m \in \mathbb{Z}_+$,

$C^m(X) =$ { m-differentiable functions on X }

For $\alpha \in (0, 1)$

$C^\alpha(X) =$ { Hölder continuous functions on X }.

$$S(X) = \left\{ f \in C^\infty(X) \;\middle|\; x^\alpha \left(\frac{d}{dx}\right)^\beta f \in L^\infty(X) \text{ for all } \alpha, \beta \in \mathbb{Z}_+ \right\}$$

$\mathcal{M}(X) = \{$ complex-valued Borel measures on X $\}$.

for $\mu \in \mathcal{M}(X)$, $\|\mu\|$ = total variation of μ

$$:= \sup_{\sqcup E_i = X} \sum_{i=1}^{\infty} |\mu(E_i)|$$

Approximate Identities.

Def: A family of functions $\{\Phi_T\}_{T=0}^{\infty} \subset L^{\infty}(\mathbb{R})$ forms an approximate identity provided that

1. $\int_{\mathbb{R}} \Phi_T = 1$ for all T

2. $\sup_T \|\Phi_T\|_{L^1} < \infty$

3. For all $\delta > 0$ one has
$$\int_{|x|>\delta} |\Phi_T(x)|\,dx \xrightarrow{T\to\infty} 0.$$

<u>Example</u>: • Let $\Phi_T := T\chi_{[-\frac{1}{T}, \frac{1}{T}]}$

• Let $\varphi \in C_c(\mathbb{R})$, $\varphi \geq 0$.

and $\Phi_T(x) = T\varphi(Tx) \cdot \frac{1}{\|\varphi\|_{L^1}}$.

<u>Prop</u>: Let $\{\Phi_T\}$ be an approximate identity. Then

i.) If $f \in C_c(X)$, then

$$\|\Phi_T * f - f\|_\infty \to 0 \quad \text{as } T \to \infty$$

ii.) If $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, then

$$\|\Phi_T * f - f\|_p \to 0 \quad \text{as } T \to \infty$$

iii.) If $\mu \in M(\mathbb{R})$, then

$$\Phi_T * \mu \xrightarrow{*} \mu \text{ as } T \to \infty$$

Pf:

(i) Note that $\int \Phi_T = 1$.

Then

$$\int \Phi_T(y) f(x-y)\,dy - f(x) \int \Phi_T(y)\,dy$$
$$= \int \Phi_T(y) \left(f(x-y) - f(x) \right)$$

Then

$$(\Phi_T * f)(x) - f(x) = \int_{|y|>\delta} \Phi_T(y) \left(f(x-y) - f(x) \right) + \int_{|y|<\delta} \Phi_T(y) \left(f(x-y) - f(x) \right)$$

$$\leq 2\|f\|_\infty \int_{|y|>\delta} |\Phi_T(y)| + \sup_{|y|<\delta} |f(x-y) - f(x)| \; \|\Phi_T\|_1$$

$$\longrightarrow 0$$

(ii.) Let $g \in C_c(\mathbb{R})$, $\|f-g\|_{L^p(\mathbb{R})} \leq \varepsilon$.

$$\|\Phi_T * f - f\|_p \leq \|\Phi_T * (f-g)\|_p + \|\Phi_T * g - g\|_p + \|f-g\|_p.$$

For T large enough.

$$\leq \|\Phi_T\|_1 \|f-g\|_p + C\|\Phi_T * g - g\|_\infty + \varepsilon.$$

(iii.) Follows easily from (i)