A Note Concerning the isometry between H and H^*.

Given H and H^*, Riesz Rep. Thm implies $\exists$ linear Banach space isometry between H and H^*. One can define an inner product on H^* by

$$\langle f,g\rangle_{H^*} := \langle \tau^{-1}(f), \tau^{-1}(g)\rangle_H.$$

So H and H^* are "the same" in a very strong sense but not always in a useful sense

Example: Consider $V \subsetneq H$, $(H,\langle\cdot,\cdot\rangle_H)$, $(V,\langle\cdot,\cdot\rangle_V)$ and V dense in H, $\langle\cdot,\cdot\rangle_V \neq \langle\cdot,\cdot\rangle_H\big|_{V\times V}$.

For example: $H = \ell^2(\mathbb{N})$ $\quad \langle a,b\rangle_H = \sum a_n \overline{b_n}$

and $\quad V = h^1(\mathbb{N}) = \{ a=(a_n)_{n=1}^\infty \mid \sum n^2|a_n|^2 < \infty\}$

with $\quad \langle a,b\rangle_V := \sum n^2 a_n \overline{b_n}$.

Then $H^* = \ell^2(\mathbb{N})$

and $\quad V^* = h^{-1}(\mathbb{N}) = \{ \sum \frac{1}{n^2} |a_n|^2 < \infty \}$.

Note: $T: V \to V^*$ isomorphism defined by $T(a) = (n^2 a_n)_{n=1}^\infty$

Cor: $\mathcal{H} \cong \mathcal{H}^*$

Def: (Orthonormal set)

The set $\{x_\alpha\}_{\alpha \in I} \subset \mathcal{H}$ is <u>orthonormal</u> if

$$\langle x_\alpha, x_\beta \rangle = \begin{cases} 0 & \alpha \neq \beta \\ 1 & \alpha = \beta \end{cases}$$

Given a linearly independent set $\{x_n\}_{n=1}^{\infty}$, one can construct an orthonormal sequence $\{e_n\}_{n=1}^{\infty}$ using Gram-Schmidt such that

$$\operatorname{span}\{e_n\}_{n=1}^{N} = \operatorname{span}\{x_n\}_{n=1}^{N} \quad \text{for all } N.$$

Bessel's Inequality

If $\{x_\alpha\}_{\alpha \in I}$ is an orthonormal set in $\mathcal{H}$, then for any $x \in \mathcal{H}$

$$\sum_{\alpha \in I} |\langle x, x_\alpha \rangle|^2 \leq \|x\|^2$$

Pf: Suffices to show for finite subcollections $E \subset A$

$$0 \leq \| x - \sum \langle x, x_\alpha \rangle x_\alpha \|^2$$
$$= \|x\|^2 - 2\mathrm{Re}\left(\sum \langle x, x_\alpha \rangle \overline{\langle x, x_\alpha \rangle}\right) + \| \sum \langle x, x_\alpha \rangle x_\alpha \|^2$$
$$= \|x\|^2 - 2 \sum |\langle x, x_\alpha \rangle|^2 + \sum |\langle x, x_\alpha \rangle|^2$$

$$\Rightarrow \quad \sum |\langle x, x_\alpha \rangle|^2 \leq \|x\|^2$$

$\square$.

Thm: If $\{x_\alpha\}_{\alpha \in I}$ is an orthonormal set in H, the following are equivalent

a.) If $\langle x, x_\alpha \rangle = 0 \ \forall \alpha \in I$, then $x = 0$.

b.) Parseval's Identity:

$$\sum_{\alpha \in I} |\langle x, x_\alpha \rangle|^2 = \|x\|^2 \qquad \forall x \in H$$

c.) For $x \in H$,

$$x = \sum_{\alpha \in I} \langle x, x_\alpha \rangle x_\alpha.$$

Pf: algebra

Def: (Orthonormal Basis)

An orthonormal set having properties (a) through (c) is called an <u>orthonormal basis</u>.

Example: $\ell^2(\mathbb{N}) = \{ a : \mathbb{N} \to \mathbb{C} \mid \sum |a_n|^2 < \infty \}$.

Let $\{ e^m_n \}_{n=1}^{\infty} = \begin{cases} 0 & n \neq m \\ 1 & n = m. \end{cases}$

then $\langle e^m, e^n \rangle = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$

and $\langle a, e^n \rangle = a_n$.

Prop: H is separable $\iff$ H has an orthonormal Schauder Basis.

Pf: ($\Rightarrow$) Use Gram-Schmidt.

($\Leftarrow$) Partial sums over $\mathbb{Q} + i\mathbb{Q}$ are dense.

Unitary Operators.

Def: Let $(H_1, \langle \cdot, \cdot \rangle_1)$ and $(H_2, \langle \cdot, \cdot \rangle_2)$ be Hilbert spaces. An invertible, linear map $U: H_1 \to H_2$ satisfying

$$\langle Ux, Uy \rangle_2 = \langle x, y \rangle_1 \quad \forall x, y \in H_1.$$

is called Unitary.

Note: U is an isometry.

Ex: Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal basis of a Hilbert space, H.

Then the map

$$\hat{\mathcal{F}} : H \to \ell^2(\mathbb{N})$$

defined by $\hat{\mathcal{F}}(x) := \{\langle x, u_n \rangle\}_{n=1}^{\infty}$ is unitary. (Pset 3).

Adjoint Operators

Let $T \in \mathcal{L}(H,H)$

Prop: a.) $\exists !\ T^* \in \mathcal{L}(H,H)$ called the **adjoint** of T satisfying

$$\langle Tx, y\rangle = \langle x, T^* y\rangle.$$

b.) $\|T^*\| = \|T\|$, $\|T^*T\| = \|T\|^2$,

$(aS + bT)^* = \bar{a} S^* + \bar{b} T^*$, $(ST)^* = T^* S^*$

and $T^{**} = T$.

c.) Let $R(T)$ = range of T

and $N(T)$ = nullspace of $T = \{x \in H \mid Tx = 0\}$.

then $R(T)^\perp = N(T^*)$

and $N(T)^\perp = \overline{R(T^*)}$.

---

pf: a.) Let $y \in H$. Define the map $f_y(x) = \langle Tx, y\rangle$.
Then $f_y \in H^*$ and by Riesz Rep thm, there exists $z_y \in H$ s.t. $\langle Tx, y\rangle = \langle x, z_y\rangle$.
Now define T^* by

$T^*: H \to H$, $T^*(y) := z_y$.

Then $\langle Tx, y \rangle = \langle x, T^* y \rangle$.

T^* is linear over $\mathbb{C}$. (Exercise).

Moreover,

$$\|T^*\| = \sup_{\substack{x,y\in H \\ \|x\|=\|y\|=1}} |\langle T^* y, x\rangle|$$

$$= \sup_{\substack{x,y\in H \\ \|x\|=\|y\|=1}} |\langle x, T^* y\rangle| = \sup_{\substack{x,y\in H \\ \|x\|=\|y\|=1}} |\langle Tx, y\rangle|$$

$$= \|T\| < \infty.$$

$\Rightarrow$ $T^* \in \mathcal{L}(H,H)$ and $\|T^*\| = \|T\|$.

b.) $$\|T^* T\| = \sup_{\|x\|=\|y\|=1} |\langle x, T^* T y\rangle|$$

$$= \sup_{\|x\|=\|y\|=1} |\langle Tx, Ty\rangle|$$

Observe $$\|T\|^2 = \sup_{\|x\|=1} \|Tx\|^2$$

$$= \sup_{\|x\|=1} |\langle Tx, Tx\rangle|$$

$$\leq \sup_{\|x\|=\|y\|=1} |\langle Tx, Ty\rangle| \leq \Big(\sup_{\|x\|=1} \|Tx\|\Big)^2$$

$$= \|T\|^2$$

$$\langle x, T^{**}y\rangle = \langle T^{*}x, y\rangle = \overline{\langle y, T^{*}x\rangle}$$
$$= \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle$$

$\Rightarrow \quad T^{**} = T.$

(c) Want to prove $R(T)^{\perp} = N(T^{*})$

and $N(T)^{\perp} = \overline{R(T^{*})}$

$y \in R(T)^{\perp} \iff \langle Tx, y\rangle = 0$ for all $x \in H$

$\iff \langle x, T^{*}y\rangle = 0$ for all $x \in H$

$\iff T^{*}y = 0 \iff y \in N(T^{*})$.

which implies $R(T)^{\perp} = N(T^{*})$.

Now, we can also conclude

$R(T^{*})^{\perp} = N(T)$.

Thus $(R(T^{*})^{\perp})^{\perp} = N(T)^{\perp}$

which implies $\overline{R(T^{*})} = N(T)^{\perp}$.

An Important Class of Banach Spaces

L^p Spaces

Let (X, M, μ) be a σ-finite, complete measure space.
For $1 \le p < \infty$, define

$$L^p(\mu) := \left\{ f : X \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ measurable} \\ f \text{ equiv. class}, \int |f|^p d\mu < \infty \end{array} \right\}.$$

$$\|f\|_{L^p(\mu)} = \|f\|_p := \left(\int_X |f|^p d\mu \right)^{1/p}.$$

The case $p = \infty$

Def: Let $f : X \to \mathbb{C}$ be measurable.
Define the <u>essential supremum</u> of f to be

$$\|f\|_\infty = \|f\|_{L^\infty(\mu)} := \inf \left\{ c \ge 0 \;\middle|\; \begin{array}{l} |f(x)| \le c \\ \text{for } \mu\text{-a.e. } x \in X \end{array} \right\}.$$

Def.

$$L^\infty(\mu) := \left\{ f : X \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ measurable}, f \text{ equiv. class} \\ \|f\|_\infty < \infty \end{array} \right\}.$$

<u>Observe:</u> $L^p(\mu)$ is a linear space for $1 \le p \le \infty$

$$|f+g|^p \le 2^p(|f|^p + |g|^p).$$

<u>Goal:</u> Show that $\|\cdot\|_p$ is a norm.

In particular,

- $\|f\|_p = 0 \iff f = 0$ (straight-forward)
- $\|\alpha f\|_p = |\alpha| \|f\|_p$ (Easy)
- Δ-inequality (More difficult).