Hilbert Spaces

Def: Let H be a complex linear space. An <u>inner product</u> (scalar product) on H is a map

$$\langle \cdot , \cdot \rangle : H \times H \to \mathbb{C}$$

satisfying

i.) $\langle \alpha x + \beta y , z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
$\forall\, x, y, z \in H$, $\alpha, \beta \in \mathbb{C}$.

ii.) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

iii.) $\langle x, x \rangle \geq 0 \quad \forall\, x \in H$. $\langle x, x \rangle > 0$ if $x \neq 0$.

(Note: $\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$.)

On a complex linear space, H, we define the function

$$\| \cdot \| : H \to [0, \infty)$$

by $$\|x\| = (\langle x, x \rangle)^{1/2}$$

Properties of the Inner Product

- (Cauchy – Schwarz)

  $$|\langle x,y\rangle| \le \|x\|\cdot\|y\|$$ (Exercise)

- (Δ - ineq)

  $$\|x+y\| \le \|x\| + \|y\|$$

<u>Proposition:</u> $\|\cdot\| : H \to [0,\infty)$ is a norm on H

<u>Def:</u> $(H, \langle\cdot,\cdot\rangle)$ is a <u>Hilbert Space</u> if H is complete with respect to the norm given by the inner product.

<u>Ex:</u> $\ell^2(\mathbb{N}) := \left\{ a = (a_n)_{n=1}^{\infty} : \mathbb{N} \to \mathbb{C} \,\middle|\, \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$.

with $\langle a,b\rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}$

Prop: Let $x, y \in H$, $\{x_n\} \subset H$, $\{y_n\} \subset H$.
If $x_n \to x$, $y_n \to y$ then

$$\langle x_n, y_n \rangle \to \langle x, y \rangle$$

Pf:

$$|\langle x, y\rangle - \langle x_n, y_n\rangle| = |\langle x - x_n, y\rangle + \langle x_n, y - y_n\rangle|$$
$$\leq |\langle x - x_n, y\rangle| + |\langle x_n, y - y_n\rangle|$$
$$\leq \|x - x_n\| \, \|y\| + \|x_n\| \cdot \|y - y_n\|$$
$$\to 0 \quad \square.$$

Def: (Parallelogram Law)
For all $x, y \in H$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Claim: Any Hilbert space norm satisfies the parallelogram law.

Also, If $(X, \|\cdot\|)$ is a real Banach space and the norm, $\|\cdot\|$, satisfies the parallelogram law then $(X, \|\cdot\|)$ is a real Hilbert space.

Def: (Orthogonality)

Let $x, y \in H$. We say that x is **orthogonal** to y, denoted $x \perp y$, if

$$\langle x, y \rangle = 0$$

Let $E \subset H$, we define

$$E^{\perp} := \{x \in H \mid \langle x, y \rangle = 0 \quad \forall y \in E\}.$$

[Note: $E^{\perp}$ is a closed linear subspace of H]

Orthogonal Projections

Thm: Suppose $M \subset H$ is a closed linear subspace of H and $x \in H$.

Then

i.) There exists an unique $y_0 \in M$ such that

$$\|x - y_0\| = \inf_{y \in M} \|x - y\|.$$

y_0 is the orthogonal projection of x to M

ii.) The element $z_0 = x - y_0$ is orthogonal to M

iii.) $H = M \oplus M^{\perp}$

pf:

(i) Let $x \in H$, $x \notin M$ and

$$A := \inf_{y \in M} \|x - y\| > 0.$$

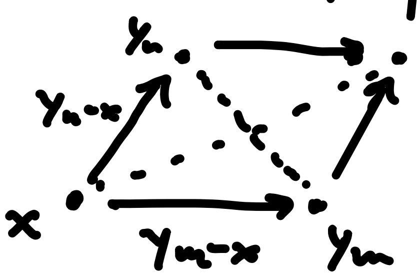
$\Rightarrow \exists \{y_n\}_{n=1}^{\infty} \subset M$ such that

$\|x - y_n\| \to A$.

Claim: $\{y_n\}_{n=1}^{\infty}$ is Cauchy.

pf of claim: Using Parallelogram Law

$$\|y_n - y_m\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - \|y_n - y_m - 2x\|^2.$$



$$= 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4\left\|\frac{y_n + y_m}{2} - x\right\|^2.$$

Observe: $y_n + y_m/2 \in M \Rightarrow \left\|\frac{y_n+y_m}{2} - x\right\|^2 \geq A^2$.

This implies

$$\|y_n - y_m\|^2 \leq 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4A^2$$

Since $\|x-y_n\| \to A$ and $\|x-y_m\| \to A$.

$\|y_n - y_m\|^2 \to 0 \Rightarrow \{y_n\}$ is Cauchy.

Completeness of H and closedness of M implies $y_0 = \lim_{n\to\infty} y_n \in M$.

and $\|x-y_0\| = A$.

Uniqueness: Let y_1, y_2 satisfy. $\|x-y_1\| = \|x-y_2\| = A$

$$\|y_1-y_2\|^2 = 2\|x-y_1\|^2 + 2\|x-y_2\|^2 - 4\left\|x - \frac{y_1+y_2}{2}\right\|^2$$

$\Rightarrow \|y_1-y_2\|^2 \le 0$.

(ii) Want to show that $z_0 = x - y_0 \in M^\perp$.

Let $w \in M$.

Define the auxilliary function

$$f(t) = \|z_0 + tu\|^2$$

where u is defined so that

$$\langle z_0, u\rangle \in \mathbb{R}$$

and $u = \xi w$ for some $\xi \in \mathbb{C}$.

By definition of z_0, $t=0$ minimizes f.

This implies $f'(0) = 0$ and

$$f'(t) = \frac{d}{dt}\left(\|z_0\|^2 + 2t\langle z_0, u\rangle + t^2\|u\|^2 \right)$$

$$= 2\langle z_0, u\rangle + 2t\,\|u\|^2$$

$$\Rightarrow \overline{\xi}\langle z_0, w\rangle = \langle z_0, u\rangle = 0 \quad \Rightarrow \quad z_0 \in M^{\perp}$$

(iii) Let $x \in H$. By part (i) and (ii), $\exists\ y_1 \in M$, $z_1 \in M^{\perp}$ such that $x = y_1 + z_1$. It remains to show that y_1 and z_1 are unique.

Suppose $x = y_2 + z_2$, $y_2 \in M$, $z_2 \in M^{\perp}$.

then $y_1 - y_2 = z_1 - z_2$.

$$\Rightarrow \quad 0 = \langle y_1 - y_2, z_1 - z_2\rangle = \|y_1 - y_2\|^2$$

$\square$.

The Dual Space of H.

Note: Let $y \in H$, define $f_y(x) = \langle x, y \rangle$ for all $x \in H$. Then $f_y \in H^* = \mathcal{L}(H, \mathbb{C})$ and $\|f_y\| = \|y\|$.

The Riesz Representation Thm.

Then Riesz Rep Thm is a class of results characterizing the dual spaces of particular Banach spaces.

Thm! (Riesz Representation Thm)
Let $F \in H^*$. There exists a unique $z \in H$ such that

$$F(x) = \langle x, z \rangle \qquad \forall x \in H.$$

Moreover, $\|F\| = \|z\|$.

Pf: Case 1: $f \equiv 0$, take $z = 0$.

Case 2: $f \not\equiv 0$

Consider $M = \{x \in H \mid f(x) = 0\}$, then

$M^\perp \neq \{0\}$.

Let $y \in M^\perp$, $\|y\| = 1$

We want to show that

$$f(x) = \langle x, \xi y\rangle \quad \text{for some } \xi \in \mathbb{C}.$$

For $x \in H$ consider $u = f(x)y - f(y)x$.

Then

$$f(u) = f(x)f(y) - f(y)f(x) = 0 \Rightarrow u \in M.$$

which further implies

$$\begin{aligned} 0 = \langle u, y\rangle &= \langle f(x)y - f(y)x, y\rangle \\ &= f(x)\|y\|^2 - f(y)\langle x, y\rangle. \\ &= f(x) - f(y)\langle x, y\rangle \end{aligned}$$

$$\Rightarrow f(x) = f(y)\langle x, y\rangle = \langle x, \overline{f(y)}\, y\rangle.$$

Let $z = \overline{f(y)}\, y$.

Exercise: Show uniqueness.