Applications of Baire Category

The Uniform Boundedness Principle

Thm: Let X be a Banach space, Y a normed linear space.

Let $A \subset \mathcal{L}(X,Y)$.

If $\forall x \in X$ and $\sup_{T \in A} \|Tx\| < \infty$.

Then $\sup_{T \in A} \|T\| < \infty$

pf: Let $E_n := \{x \in X \mid \sup_{T \in A} \|Tx\| \leq n\}$.

$$= \bigcap_{T \in A} \{x \in X \mid \|Tx\| \leq n\}.$$

For each $T \in A$, $\{x \in X \mid \|Tx\| \leq n\}$ is closed

$\Rightarrow$ E_n is closed.

Observe:
- $E_n \subset E_{n+1}$
- $X = \bigcup_{n=1}^{\infty} E_n$

Baire Category implies that since X is a Banach space, at least one of the E_n is not nowhere dense.

$$\Rightarrow \operatorname{int}(E_n) = \operatorname{int}(\overline{E_n}) \neq \emptyset.$$

$$\Rightarrow \exists B(x_0, r_0) \subset \overline{B(x_0, r_0)} \subset \operatorname{int}(E_n) \text{ for some } n.$$

$$\Rightarrow \text{for all } T \in A, \sup_{y \in \overline{B(0,r_0)}} \|T(x_0+y)\| \leq n.$$

$$\Rightarrow \sup_{T \in A} \sup_{y \in \overline{B(0,r_0)}} \|Tx_0 + Ty\| \leq n$$

$$\Rightarrow \sup_{T \in A} \sup_{y \in \overline{B(0,r_0)}} \|Ty\| \leq n + \|Tx_0\|$$

$$\Rightarrow \sup_{T \in B} \sup_{\|y\| \leq 1} \|y\| \leq \frac{2n}{r_0}.$$

$\square$

Applications of Baire Category.

Open Mapping Theorem

Def: Let X, Y be top. spaces. A function $f: X \to Y$ is <u>open</u> if for all open $U \subset X$, $f(U)$ is open in Y.

Important Example of open function

Let X, Y be normed linear spaces, and consider $X \times Y$ with the norm

$$\|(x,y)\| := \|x\|_X + \|y\|_Y$$

The projection $\pi: X \times Y \to X$ defined by

$$\pi(x,y) = x$$

is open.

Lemma: Let X, Y be normed linear spaces and $T: X \to Y$ be a linear transformation. T is open iff $\exists r > 0$ s.t.

$$B^Y(0,1) \subset T(B^X(0,r))$$

Pf: Exercise

Thm: (Open Mapping Thm)

Let X, Y be Banach Spaces. If $T \in \mathcal{L}(X,Y)$ is surjective, then T is open.

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pf: T is surjective which implies

$$T(X) = Y \quad \Rightarrow \quad \bigcup_{n=1}^{\infty} T(B^X(0,n)) = T.$$

Observe: By Lemma, it suffices to show that $\exists r > 0$ s.t. $B(0,r) \subset T(B(0,n))$ for some n

Baire Category implies that for some $n \geq 1$, $T(B^X(0,n))$ is not nowhere dense

$$\Rightarrow \exists y_0 \in \operatorname{int}\left(\overline{T(B(0,n))}\right), \ \exists r > 0 \ \text{s.t.}$$

$$B^Y(y_0, r) \subset \overline{T(B^X(0,n))}.$$

Claim: $B^Y(0, r/100) \subset T(B^X(0, 4n))$.

Proof of claim:

Let $y \in B^Y(0, r)$

Then $y + y_0 \in B^Y(y_0, r) \subset \overline{T(B(0,n))}$

Let $\varepsilon > 0$, $\exists z_1, z_2 \in T(B^X(0,n))$
such that $z_i = Tx_i$ for some $x_i \in B(0,n)$
and $\|y_0 - z_1\| < \varepsilon/2$ and $\|y + y_0 - z_2\| < \varepsilon/2$

Then $\|(z_1 - z_2) - y\| \leq \|y_0 - z_1\| + \|y + y_0 - z_2\| < \varepsilon$

and $z_1 - z_2 = Tx_1 - Tx_2 = T(x_1 - x_2)$.

Note: $x_1 - x_2 \in B^X(0, 2n)$

Thus $B^Y(0, r) \subset \overline{T(B^X(0, 2n))}$

We observe that linearity implies

$$B^Y(0, r/2^k) \subset \overline{T(B^X(0, \tfrac{2n}{2^k}))}$$

Then if $y \in B^Y(0,r)$, $\exists x_1 \in B^X(0,2n)$
such that $\|y - Tx_1\| < r/2$

Now $y - Tx_1 \in B^Y(0, r/2)$
which implies that there exists
$x_2 \in B^X(0, \frac{2n}{2})$ s.t. $\|y - Tx_1 - Tx_2\| < r/4$

By induction, construct a sequence $\{x_k\}_{k=1}^{\infty}$ such that

$\|x_k\| < 2n \cdot 2^{-k}$ and $y = \sum_{k=1}^{\infty} T x_k$.

X complete implies $\exists x$ such that

$$\sum_{k=1}^{\infty} x_k = x \in B^X(0, 4n)$$

Then T continuous implies that $Tx = y$ and therefore

$$B^Y(0, r/2) \subset T(B(0, 4n))$$

$\square$.

Corollary: A test for invertibility

Cor: If X and Y are Banach spaces, and $T \in \mathcal{L}(X, Y)$ is bijective, then T is an isomorphism

Closed Graph Thm

Def: (Closed/Graph)

① X, Y normed linear spaces and $T: X \to Y$ linear. The graph of T is defined as

$$\text{Graph}(T) = \{(x, Tx) \mid x \in X\}$$

Note: Graph(T) is a linear subspace of $X \times Y$.

② T is closed if Graph(T) is a closed subset of $(X \times Y, \|\cdot\|_{X\times Y})$

where $\|(x,y)\|_{X\times Y} := \max(\|x\|_X, \|y\|_Y)$.

<u>Thm:</u> (Closed Graph Thm)

Let X, Y be Banach Spaces. If $T: X \to Y$ is a closed linear map, then T is continuous.

<u>Pf:</u> Let

$$\pi_1 : \text{Graph}(T) \to X$$

$$\pi_2 : \text{Graph}(T) \to Y$$

be projections defined by

$$\pi_1((x, Tx)) = x$$

$$\pi_2((x, Tx)) = Tx$$

Observe: $\pi_1 \in \mathcal{L}(\text{Graph}(T), X)$ and $\pi_2 \in \mathcal{L}(\text{Graph}(T), Y)$

Moreover, π_1 is surjective which, by the open mapping thm, implies that $\pi_1^{-1} \in \mathcal{L}(X, \text{Graph}(T))$.

Therefore,

$$T = \pi_2 \circ \pi_1^{-1} \Rightarrow T \text{ is a composition}$$

of continuous functions

$\square$

Corollary: A test for continuity

It suffices to show that if $\{x_n\}_{n=1}^{\infty}$ and $\{Tx_n\}_{n=1}^{\infty}$ both converge then

$$\lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n),$$

by the Closed mapping thm. Without Closed mapping, one assumes $\{x_n\}$ converges but then must show that $\{Tx_n\}$ converges and converges to the correct thing: $T(\lim_{n\to\infty} x_n)$.