A Bit of Point Set Topology.

Def: (Topology)

A **topology** on a set, X, is a collection, $\tau \subset \mathcal{P}(X)$, of subsets of X having the following properties:

a.) $\emptyset \in \tau$, $X \in \tau$

b.) $\forall \mathcal{F} \subset \tau$, $\bigcup_{G \in \mathcal{F}} G \in \tau$

c.) For any $\{G_1, \ldots, G_n\}$, $\bigcap_{i=1}^{n} G_i \in \tau$

The pair (X, τ) is called a **topological space**

Def: (Basis of a Topology)

If X is a set, a **basis** for a topology on X is a collection, $\mathcal{B}$, of subsets of X (called **basis elements**) such that a.) For each $x \in X$, $\exists B \in \mathcal{B}$ such that $x \in B$

b.) If $x \in B_1 \cap B_2$, for $B_1, B_2 \in \mathcal{B}$ then $\exists B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subset B_1 \cap B_2 .$$

<u>Def:</u> (Topology generated by a Basis)

Given a basis, $\mathcal{B}$, of X. The topology, τ, generated by $\mathcal{B}$ is defined by

$$U \in \tau \iff \text{For each } x \in U, \ \exists B \in \mathcal{B} \text{ such that } x \in B \subset U.$$

<u>Example:</u> Very weak/coarse topology

$$\tau = \{\phi, X\}.$$

<u>Example:</u> very strong/fine topology

$$\tau = \mathcal{P}(X)$$

Example: Given a normed space, $(X, \|\cdot\|)$, define the basis

$$\mathcal{B} = \{ B(x,r) \mid x \in X, r > 0 \}.$$

then the topology generated by $\mathcal{B}$ is the norm/strong topology.

Terminology

- Consider two topologies, τ and τ'. If $\tau \subset \tau'$ then τ is said to be weaker than τ' and τ' is said to be stronger than τ.
- Weak Topologies.

  Consider a collection of continuous real-valued functions,

  $$\mathcal{F} \subset \{ f : X \to \mathbb{R} \mid f \text{ continuous} \}.$$

  on a topological space, X. We define the $\mathcal{F}$-weak topology as the weakest topology for which every function in $\mathcal{F}$ is continuous.

Lemma: Let $\{x_n\}_{n=1}^{\infty} \subset X$, $\mathcal{F}$ a collection of continuous real-valued functions.

Then

$x_n \to x$ in the $\mathcal{F}$-weak topology $\iff f(x_n) \to f(x)$ $\forall f \in \mathcal{F}$.

Def: (The Weak Topology)

Let $(X, \|\cdot\|)$ be a normed linear space. The **weak topology** of $(X, \|\cdot\|)$ is defined as the

X^*-weak topology of $(X, \|\cdot\|)$

Def: (The Weak-* Topology)

Let $(X, \|\cdot\|)$ be a normed linear space. The **weak-* topology on X^*** is defined as the

$\hat{X}$-weak topology of $(X, \|\cdot\|)$.

Lemma:

weak-* topology on X^*

$\subset$ weak topology on X^*

$\subset$ strong topology on X^*

Baire Category Theorem and Applications.

Def: (Nowhere dense)

If (X,τ) is a top. space, $A \subset X$ is <u>nowhere dense</u> if $\mathrm{int}(\overline{A}) = \emptyset$.

Ex: Sets of isolated points

Ex: Cantor sets.

Lemma: E nowhere dense

$$\Longleftrightarrow (\overline{E})^c \text{ is open and dense.}$$

Def: (Meager / First Category)

$A \subset X$ is <u>meager</u> (First category) in X if A is a countable union of nowhere dense sets

Def: (Second Category)

If A is not meager then it is of <u>second category</u>

Def: (Generic sets)

The complement of a meager set is a generic set (<u>residual set</u>).

Thm (Baire Category Thm)

Let X be a complete metric space.

① If $\{U_n\}_{n=1}^{\infty}$ is a sequence of open dense subsets of X, then

$$\bigcap_{n=1}^{\infty} U_n \text{ is dense in } X$$

② X is not a countable union of nowhere dense sets.

Proof:

1 Goal: Show that for every open set $V \subset X$,

$$V \cap \left(\bigcap_{n=1}^{\infty} U_n \right) \neq \emptyset$$

Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of open dense subsets of X. Then U_1 is dense in X, which implies that for every V open, $V \cap U_1$ is open and

$$V \cap U_1 \neq \emptyset .$$

$$\Rightarrow \quad \exists\, B(x_1, r_1) \subset \overline{B(x_1, r_1)} \subset V \cap U_1$$

such that $r_1 < 1/2$.

Furthermore, $U_2 \cap B(x_1, r_1) \neq \emptyset$ and is open.

Thus, $\exists\, B(x_2, r_2) \subset \overline{B(x_2, r_2)} \subset U_2 \cap B(x_1, r_1)$
$\subset V \cap (U_1 \cap U_2)$

Inductively define

$$B(x_k, r_k) \subset \overline{B(x_k, r_k)} \subset V \cap \left(\bigcap_{n=1}^{k} U_n \right)$$

$\Rightarrow \quad \{x_k\}_{k=1}^{\infty}$ is Cauchy

X complete $\Rightarrow \quad x_k \to x \in \overline{B(x_n, r_n)} \quad \forall n.$

$$\Rightarrow \quad x \in V \cap \left(\bigcap_{n=1}^{\infty} U_n \right)$$

② Let E_n be nowhere dense for all $n \geq 1$.

It suffices to show that

$$\left(\bigcup_{n=1}^{\infty} \overline{E_n} \right)^c \neq \phi.$$

Thus, $\left(\bigcup_{n=1}^{\infty} \overline{E_n} \right)^c = \bigcap_{n=1}^{\infty} \overline{E_n}^c$

Note, $(\overline{E_n})^c$ is open and dense.

part ① $\Rightarrow \bigcap_{n=1}^{\infty} (\overline{E}^c) \neq \phi \quad \square.$