

Corollaries

Cor: Let $M \subset X$ be a linear subspace.

$\Rightarrow$ $f \in M^*$, then there exists $F \in X^*$

s.t. $F|_M = f$ and

$$\|F\|_{X^*} = \|f\|_{M^*}$$

γf : Let $p(x) = \|f\|_{M^*} \|x\|$

Note: p is a sublinear functional

Hahn-Banach $\Rightarrow \exists F: X \rightarrow \mathbb{R}$, linear
such that $F|_M = f$ and

$$|F(x)| \leq p(x)$$

$$\Rightarrow |F(x)| \leq \|f\|_{M^*} \|x\| \Rightarrow \|F\|_{X^*} \leq \|f\|_{M^*}$$

and

$$\|f\|_{M^*}$$

$$= \sup_{\substack{x \in M \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x \in M \\ x \neq 0}} \frac{|F(x)|}{\|x\|}$$

$$\leq \sup_{\substack{x \in X \\ x \neq 0}} \frac{|F(x)|}{\|x\|} \leq \|F\|_{X^*}$$

$$\Rightarrow \|F\|_{X^*} = \|f\|_{M^*}$$

$\square$.

Q: Is f unique?

A: No, not necessarily

Example: Consider $X = \ell^2(\mathbb{N})$

Define $f(\{a_n\}_{n=1}^{\infty}) = a_1$

Then $f \in (\ell^2)^*$, $\|f\| = 1$.

For $\alpha \in [-1, 1]$ and define

$$g_\alpha : (\ell^2) \rightarrow \mathbb{R} \text{ by} \\ g_\alpha(\{a_n\}_{n=1}^{\infty}) = a_1 + \alpha a_2$$

$$\begin{aligned} \text{Then } |g_\alpha(\{a_n\})| &\leq |a_1| + |\alpha| |a_2| \\ &\leq \|\{a_n\}\|_{\ell^2} \end{aligned}$$

Cor: If $M \subset X$ is a closed, linear subspace and $x \in X \setminus M$, then $\exists f \in X^*$ such that

- $f(x) \neq 0$
- $f|_M = 0$
- $\|f\|_{X^*} = 1$.

Pf: Since M is a closed, linear subspace we can define

$$0 < \delta := \inf_{y \in M} \|x - y\| \leq \|x\|.$$

Define

$$g: (M + \mathbb{R}x) \rightarrow \mathbb{R}$$

$$\text{by } g(y + \lambda x) := \lambda \delta \quad \forall y \in M.$$

Then

$$g|_M = 0 \quad \text{and} \quad g(x) = \delta \neq 0$$

Let

$$p(z) := \|z\|. \quad \text{Since } \delta \leq \|x\|, |g(z)| \leq p(z).$$

Hahn-Banach implies $\exists f: X \rightarrow \mathbb{R}$ linear functional such that

$$f|_M = g|_M = 0$$

and

$$|f(z)| \leq p(z) \leq \|z\| \Rightarrow \|f\|_{X^*} \leq 1.$$

□.

Exercise: Show that one can define f so that $\|f\|_{X^*} = 1$.

Dual Elements

Idea: Construct correspondence between elements in X and elements in X^* .

Given a normed, linear space $(X, \|\cdot\|)$
for any $y \in X$, define

$$F(y) = \left\{ f \in X^* \mid \|f\| = \|y\| \text{ and } f(y) = \|y\|^2 \right\}$$

Prop: Let $(X, \|\cdot\|)$ be a normed, linear space, then for each $x \in X$, $F(x)$ is closed, convex and nonempty.

Pf: ① $F(x)$ is nonempty by Hahn-Banach.

② $F(x) = \{ f(x) = \|x\|^2 \} \cap \{ \|f\| = \|x\| \}.$

$\{ \|f\| = \|x\| \}$ is closed

and $|f(x) - f_n(x)| \leq \|x\| \|f_n - f\| \rightarrow 0.$

for $\{f_n\} \subset F(x)$, $\|f_n - f\| \rightarrow 0.$

③ $F(x)$ is convex

If $f, g \in F(x)$,

$$\begin{aligned}\|t f + (1-t) g\| &\leq t \|f\| + (1-t) \|g\| = t \|x\| + (1-t) \|x\| \\ &= \|x\|\end{aligned}$$

and

$$\begin{aligned}(t f + (1-t) g)(x) &= t f(x) + (1-t) g(x) \\ &= t \|x\|^2 + (1-t) \|x\|^2 = \|x\|^2.\end{aligned}$$

$$\Rightarrow \|t f + (1-t) g\| \geq \|x\|$$

thus $\|t f + (1-t) g\| = \|x\|$ and $t f + (1-t) g \in F(x)$.