A note on unbounded linear functions:

Q: How does one prove the existence of unbounded linear operators?

A: Consider $X = \ell^\infty(\mathbb{N})$, $Y = \mathbb{R}$.

By Part 1, $\exists$ Hamel basis, $\{e_\alpha\}_{\alpha \in I}$, of X. Let $\{f_k\}_{k=1}^{\infty} \subset \{e_\alpha\}_{\alpha\in I}$ be a countably infinite subset of $\{e_\alpha\}_{\alpha \in I}$.

Define
$$T(e_\alpha) = \begin{cases} 2^k \|f_k\| & e_\alpha = f_k \\ 1 & e_\alpha \neq f_k. \end{cases}$$

For every, $x \in X$, $\exists \{\lambda_{\alpha_i}\}_{i=1}^N$, such that $x = \sum_{i=1}^N \lambda_{\alpha_i} e_{\alpha_i}$ define $T(x) := \sum_{i=1}^N \lambda_{\alpha_i} T e_{\alpha_i}$

Then $T : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ is linear, but
$$\frac{|T(f_k)|}{\|f_k\|} \geq 2^k .$$

Notes: (1) If $T \in \mathcal{L}(X,Y)$, $S \in \mathcal{L}(Y,Z)$ then $S \circ T = ST \in \mathcal{L}(X,Z)$

(In particular, $\mathcal{L}(X,X)$ is an algebra)

(2) $T \in \mathcal{L}(X,Y)$ is invertible if T is a bijection and T^{-1} is a bounded linear map.

(3) T is an isometry if $\|Tx\|_Y = \|x\|_X$ for all $x \in X$.

An Important Class of Operators: The Dual Space

Def: Let X be a normed linear space. The dual space of X, denoted X^*, is defined as

$$X^* = \mathcal{L}(X, \mathbb{R})$$

(Elements in X^* are called linear functionals)

Cor: X^* is a Banach space.

Example: $(\mathbb{R}^d)^* \cong \mathbb{R}^d$ (Part 1).

Notation: For $x \in X$, $f \in X^*$, we, at times, denote $f(x)$ as a pairing of f and x

$$\langle f, x \rangle := f(x)$$

The Double Dual.

Given a normed linear space, $(X, \|\cdot\|)$, and its dual, X^*, define

$$X^{**} = (X^*)^*$$

Elements in X^{**}

Let $x \in X$, define $\hat{x} : X^* \to \mathbb{R}$

by $\hat{x}(f) := \langle f, x \rangle = f(x)$.

Claim: $\hat{x} \in X^{**}$ (Observe: $|\hat{x}(f)| = |f(x)| \leq \|f\| \cdot \|x\|$.)

The map $x \mapsto \hat{x}$ is called the <u>natural embedding</u> of X into X^{**}

<u>Exercise</u>: $x \mapsto \hat{x}$ is an isometry.

<u>Q</u>: In general, does X^* contain nontrivial elements?

<u>A</u>: Yes, we will construct many using axiom of choice.

Important Statements

① <u>Hausdorff Maximal Principle:</u>

Every partially ordered set has a maximal (linearly / ~~totally~~) ordered subset

② <u>Zorn's Lemma</u>

If X is a partially ordered set and every linearly ordered subset of X has an upper bound, then X has a maximal element

Def: (Sublinear Functional)

Let X be a linear space. A sublinear functional on X is a map $p: X \to \mathbb{R}$ such that

- $p(x+y) \leq p(x) + p(y) \quad \forall x, y \in X$
- $p(\lambda x) \leq \lambda p(x)$ for $\lambda > 0$, $x \in X$.

Geometric Intuition

Linear functional on $\mathbb{R}^d$ $\longleftrightarrow$ hyperplane through the origin in $\mathbb{R}^{d+1}$

sublinear functional on $\mathbb{R}^d$ $\longleftrightarrow$ upward cone w/ vertex at the origin in $\mathbb{R}^{d+1}$.

Thm: (Hahn-Banach)

Let X be a real linear space, p a sublinear functional on X, M a subspace of X, and f a linear functional on M such that $f(x) \leq p(x)$ $\forall x \in M$. Then there exists a linear functional, F, on X s.t. $F(x) \leq p(x)$ $\forall x \in X$ and $F|_M = f$.

proof:

Plan: Show that we can extend f by one dimension, then use Hausdorff Maximum Principle to choose a maximal extension

Let $f: M \to \mathbb{R}$, $f(x) \le p(x)$ for $x \in M$

Let $x_0 \in X \setminus M$ and $\mathbb{R}x_0 := \{ t x_0 \in X \mid t \in \mathbb{R} \}$.

We want to show $\exists g: (M + \mathbb{R}x_0) \to \mathbb{R}$ s.t. g is linear, $g|_M = f$ and $g \le p$ on $M + \mathbb{R}x_0$

Aside: A Bad Strategy

Let $\beta = p(x_0)$ and define

$$g: (M + \mathbb{R}x_0) \to \mathbb{R} \quad \text{by}$$

$$g(x + \lambda x_0) = f(x) + \lambda \beta$$

Then g is linear, $g(\lambda x_0) \le p(\lambda x_0)$

and $g(x) \le p(x)$ for all $x \in M$

But $g(x + \lambda x_0)$ is not necessarily less than $p(x + \lambda x_0)$

Example: Define $f: \mathbb{R}\times\{0\} \subset \mathbb{R}^2 \to \mathbb{R}$ by $f(x,0)=x$. Define $p(x,y) = (x^2+y^2)^{1/2}$.

Then $f(x,0) \leq p(x,0)$.

Define $g(x,y) := x+y$, then

$g(x,0) = f(x,0) \leq p(x,0)$ and $g(0,y) \leq p(0,y)$

but $g(1,1) = 2 > p(1,1) = \sqrt{2}$.

Continuing the actual proof:

Observe for $y_1, y_2 \in M$

$$f(y_1) + f(y_2) = f(y_1+y_2) \leq p(y_1+y_2) \leq p(y_1 - x_0) + p(x_0+y_2)$$

$$\Rightarrow \quad f(y_1) - p(y_1 - x_0) \leq p(x_0+y_2) - f(y_2).$$

$$\Rightarrow \quad \sup_{y \in M} f(y) - p(y-x_0) \leq \inf_{y \in M} p(x_0+y) - f(y).$$

Let α satisfy

$$\sup_{y \in M} f(y) - p(y-x_0) \leq \alpha \leq \inf_{y \in M} p(x_0+y) - f(y).$$

Now define $g: (M + \mathbb{R}x_0) \to \mathbb{R}$ by $g(y+\lambda x_0) = f(y) + \lambda\alpha$.

Claim: g is linear, $g|_M = f$ and

$$g(y+\lambda x_0) \le p(y+\lambda x_0) \quad \forall y \in M, \lambda \in \mathbb{R}$$

Proof of claim: In cases

1. $\lambda = 0$, $g(y) = f(y) \le p(y)$ by assumption
2. $\lambda > 0$

$$g(y+\lambda x_0) = f(y) + \lambda \alpha = \lambda \left(f(y/\lambda) + \alpha \right)$$

$$\le \lambda \left(f(y/\lambda) + p(x_0 + y/\lambda) - f(y/\lambda) \right)$$

$$= p(y+\lambda x_0).$$

3. $\lambda < 0$ similar.

Next Step: Consider the class

$$\mathcal{F} = \left\{ g : N \to \mathbb{R} \;\middle|\; \begin{array}{l} M \subset N \subset X \\ \text{linear subspace} \end{array} \quad g|_M = f, \; g \le p \text{ on } N \right\}$$

$\ne \emptyset$.

$\mathcal{F}$ is a partially ordered set:

If $g_1 : N_1 \to \mathbb{R}$, $g_2 : N_2 \to \mathbb{R}$ belong to $\mathcal{F}$

$$g_1 < g_2 \iff N_1 \subset N_2 \text{ and } g_2 = g_1 \text{ on } N_1.$$

Hausdorff Max. Princ.

$\Rightarrow$ $\exists$ maximal linearly (totally) ordered family

$\{g_\alpha\}_{\alpha \in I} \subset \mathcal{F}$, $g_\alpha : N_\alpha \to \mathbb{R}$.

Define $$F : \bigcup_\alpha N_\alpha \to \mathbb{R}$$

$$F(x) := g_\alpha(x) \quad \text{if } x \in N_\alpha.$$

<u>Note:</u>

- F is well-defined since $g_{\alpha_2} = g_{\alpha_2}\big|_{N_{\alpha_1}}$ when $g_{\alpha_1} < g_{\alpha_2}$.
- $\bigcup_\alpha N_\alpha$ is a linear space
- $\bigcup_\alpha N_\alpha = X$ since $\{g_\alpha\}_{\alpha \in I}$ is maximal
- F is linear
- $F \leq p$

$\square$.