(2) A linear map, $T: X \to Y$, is <u>continuous</u> if $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$\|x-y\|_X < \delta \Rightarrow \|Tx - Ty\| < \varepsilon.$$

## Examples

(a) Let $X = L^1(\mu)$, $Y = \mathbb{R}$.

Show that $T: L^1(\mu) \to \mathbb{R}$ defined by

$$T(f) := 10 \int f \, d\mu$$

is a bounded, linear map.

(b) Let $X = L^1(\mathbb{R})$, $Y = \mathbb{R}$

consider, $E \subset \mathbb{R}$, $F \subset \mathbb{R}$, $E \cap F = \phi$

and

$$g = 2\chi_E + 3\chi_F$$

Show that $T(f) := \int f g \, dm$

is a bounded linear map.

(c) Consider $X = \ell^1(\mathbb{N})$

Define the operator

$$T(\{a_n\}_{n=1}^{\infty}) := \{n a_n\}_{n=1}^{\infty}$$

Is $T : \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$? No

Is $T : \ell^1(\mathbb{N}) \to \ell^\infty(\mathbb{N})$? NO.

Is $T$ bounded? continuous? No

Consider $A_m := \{a_n^m\}_{n=1}^{\infty}$

where $a_n^m = \begin{cases} 0 & n \neq m \\ m^{-1/2} & n = m \end{cases}$

$$\Rightarrow \quad \|A_m\|_{\ell^1(\mathbb{N})} = \sum |a_n^m| = m^{-1/2}.$$

$$\Rightarrow \quad A_m \xrightarrow{m\to\infty} 0$$

$T(A_m) = \{n a_n^m\}_{n=1}^{\infty}$ and $n a_n^m = \begin{cases} 0 & n \neq m \\ m^{1/2} & n = m \end{cases}$

$$\Rightarrow \quad \|T(A_m)\|_{\ell^\infty(\mathbb{N})} = m^{1/2} \xrightarrow{m\to\infty} \infty.$$

$\Rightarrow$ $T$ is not bounded or continuous.

(d) Consider $X = C^1([-1,1])$

$$= \left\{ f:[-1,1]\to\mathbb{R} \;\middle|\; \begin{array}{l} f \text{ continuously} \\ \text{differentiable on} \\ [-1,1] \end{array} \right\}.$$

Define the map

$$T: C^1([-1,1]) \to C([-1,1]).$$

by $T(f) := f'$

Suppose $C^1([-1,1])$. is equipped the supremum norm, $\|\cdot\|_\infty$.

Is $T$ bounded? continuous? No

Let $g_n(x) = \frac{1}{n^{1/2}} \exp(-n x^2)\, \chi_{[-1,1]}(x)$

Exercise: Show that $\|g_n\|_\infty \to 0$

but $\|T g_n\|_\infty \to \infty$.

# More Linear Operators.

If $X$ and $Y$ are normed linear spaces we denote

$$\mathcal{L}(X,Y) := \left\{ T : X \to Y \;\middle|\; \begin{array}{l} T \text{ is linear} \\ T \text{ is bounded} \end{array} \right\}.$$

Observation: $\mathcal{L}(X,Y)$ is a linear space.

**Proposition:**

Let $X, Y$ be normed linear spaces and $T : X \to Y$ be a linear map. The following are equivalent

1. $T$ is continuous
2. $T$ is continuous at $0$
3. $T$ is bounded

$\boxed{\text{Proof}}$ $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

# Structure of $\mathcal{L}(X,Y)$

For $T \in \mathcal{L}(X,Y)$ define the operator norm of $T$ by

$$\|T\| := \sup\{\|Tx\| : \|x\|=1\}.$$

$$:= \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\}.$$

$$:= \inf\{c > 0 \mid \|Tx\| \leq c\|x\| \ \forall x \in X\}.$$

**Note:** Given these definitions,

$$\|Tx\|_Y \leq \|T\| \, \|x\|_X$$

**Exercise:** All three definitions are equivalent

**Exercise:** $\|T\|$ is a norm on $\mathcal{L}(X,Y)$.

Proposition: If $Y$ is a Banach space, then $\mathcal{L}(X,Y)$ (equipped with the operator norm) is a Banach space.

Pf: Goal: $Y$ complete $\Rightarrow \mathcal{L}(X,Y)$ complete.

Let $\{T_n\}_{n=1}^{\infty} \subset \mathcal{L}(X,Y)$ be a Cauchy sequence.

For each $x \in X$

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\|$$
$$\leq \|T_n - T_m\| \, \|x\|.$$

$\Rightarrow \{T_n x\}_{n=1}^{\infty} \subset Y$ is a Cauchy sequence.

Since $Y$ is complete, for all $x \in X$, $\exists\, z_x \in Y$ s.t. $T_n x \xrightarrow{n\to\infty} z_x$

Define the function

$$Tx := z_x = \lim_{n\to\infty} T_n x.$$

Claim: $T \in \mathcal{L}(X,Y)$ and $\|T_n - T\| \to 0$.

① $T \in \mathcal{L}(X,Y)$

T is linear: $T_n$ are linear and limits are linear.

It remains to show that T is bounded.

Let $x \in X$, then

$$\|Tx\| \leq \|(T-T_n)x\| + \|T_n x\|$$

$$\leq \|(T-T_n)x\| + \|T_n\| \|x\|$$

$$\leq \underbrace{\|(T-T_n)x\|}_{\to 0} + \left(\sup_n \|T_n\|\right) \|x\|$$

and $\sup_n \|T_n\| < \infty$.

$$\Rightarrow \quad \|Tx\| \leq \left(\sup_n \|T_n\|\right) \|x\|$$

$$\Rightarrow \quad \|T\| \leq \left(\sup_n \|T_n\|\right).$$

$\Rightarrow$ T is bounded.

## ② $\|T - T_n\| \to 0$

Let $\varepsilon > 0$, $\{T_n\}$ Cauchy implies $\exists N \in \mathbb{N}$ s.t. if $n, m \geq N$ then

$$\|T_n - T_m\| < \varepsilon/100.$$

Claim: For any $n \geq N$, $\exists x_n \in X$ such that $\|x_n\| = 1$ and

$$\|T - T_n\| \leq \|(T - T_n)x_n\| + \varepsilon/100.$$

Assuming the claim holds, we also note that for any fixed $n \geq N$, $\exists M$ s.t. $m \geq M$ implies

$$\|Tx_n - T_m x_n\| < \varepsilon/100$$

Now for $n \geq N$, $m \geq \max(N, M)$

$$\begin{aligned}
\|T_n - T\| &\leq \|Tx_n - T_n x_n\| + \varepsilon/100 \\
&\leq \|Tx_n - T_m x_n\| + \|T_m x_n - T_n x_n\| + \varepsilon/100 \\
&\leq \varepsilon/100 + \|T_m - T_n\| \|x_n\| + \varepsilon/100 \\
&< \varepsilon.
\end{aligned}$$

$\square$