Cor: If $\sigma(T)=\{0\}$ and T is self-adjoint, then $T=0$.

Thm: (Spectral Decomposition)
Let $(H, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space. Let $T \in K(H)$ be self-adjoint. Then there exists an orthonormal Schauder basis composed of eigenvectors of T.

Pf:
Let $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be the sequence of all distinct nonzero eigenvalues of T.

Let $\lambda_0 = 0$, $E_0 = N(T)$.

and $E_n = N(T-\lambda_n I)$ (Note: $0<\dim(E_n)<\infty$)

Claim: $H = \overline{\bigoplus_{n=0}^{\infty} E_n}$

part 1 The spaces, E_n, are mutually orthogonal

If $u \in E_n$, $v \in E_m$ $m \neq n$, then

$Tu = \lambda_n u$ and $Tv = \lambda_m u$

$\Rightarrow \lambda_n \langle u, v \rangle = \langle Tu, v \rangle = \langle u, Tv \rangle = \lambda_m \langle u, v \rangle$

$\Rightarrow \langle u, v \rangle = 0.$

part 2 $\sum_{n=0}^{\infty} E_n$ is dense in H.

Let

$F := \{ f \in H \mid \exists M \in \mathbb{Z}_+, u_n \in E_n, \text{ s.t. } f = \sum_{n=0}^{M} u_n \}.$

Observe: $T(F) \subset F$

Then if $u \in F^{\perp}$, $\langle Tu, v \rangle = \langle u, Tv \rangle = 0$

for all $v \in F$

$\Rightarrow Tu \in F^{\perp} \Rightarrow T(F^{\perp}) \subset F^{\perp}$

Define $$T_0 := T|_{F^\perp}$$

Then T_0 is a self-adjoint, compact operator

Claim: $\sigma(T_0) = \{0\}$.

pf of claim: If not, $\exists \lambda \neq 0$ such that

$\lambda \in \sigma(T_0) \Rightarrow \lambda \in EV(T_0)$.

This implies there exists $u \in F^\perp$, $u \neq 0$ such that

$$T_0 u = \lambda u$$

and $$Tu = \lambda u$$

$\Rightarrow \lambda \in EV(T)$, and $\lambda = \lambda_n$ for some $n \neq 0$, this implies $u \in F$.

$\Rightarrow u \in F \cap F^\perp$

$\Rightarrow u = 0$ $\lightning$.

By previous corollary, $T_0 = 0$

$$\Rightarrow T|_{F^\perp} = 0$$

Finally, $$F^\perp \subset E_0 \subset F$$

and therefore, $F^\perp = \{0\}$.

$$\Rightarrow \quad \overline{F} = (F^\perp)^\perp = H \quad \Rightarrow F \text{ is dense in } H.$$

Finally, for each E_n, $n \geq 0$, one can construct an orthonormal basis, $\{e_m^n\}$,

Then $\bigcup_{n\geq 0} \{e_m^n\}_m$ is an orthonormal basis of H and

$$Te_m^n = \lambda_n e_m^n \qquad \square.$$

The Hille-Yosida Theorem

Def: An unbounded operator,

$$T : D(T) \subset H \to H$$

is said to be monotone (accretive) if.

$$\langle Tu, u \rangle \geq 0 \qquad \forall u \in D(T)$$

It is called maximal monotone if, in addition,

$$R(I+T) = H.$$

Prop: Let T be a maximal monotone operator. Then

a.) $D(T)$ is dense in H.

b.) T is a closed operator.

c.) For every $\lambda < 0$

$$(T - \lambda I)^{-1} \in \mathcal{L}(H, H)$$

and $\quad \|(T - \lambda I)^{-1}\| \leq 1.$

Pf. ⓐ Let $f \in H$, Since $R(I+T) = H$

$\exists v \in D(T)$ s.t. $f = v + Tv$.

Then $\langle f, v \rangle = |v|^2 + \langle Tv, v \rangle \geq |v|^2$.

Thus, $\langle f, u \rangle = 0 \quad \forall u \in D(T) \Rightarrow f = 0$

$\Rightarrow D(T)$ is dense

b) First, for any $f \in H$, $\exists! v \in D(T)$

s.t. $f = v + Tv$. Otherwise, if

$f = v' + Tv'$, then

$v' - v + Tv' - Tv = 0$

$\Rightarrow 0 = \|v' - v\|^2 + \langle T(v'-v), v'-v \rangle \geq \|v'-v\|^2$

$\Rightarrow v' = v$

Let $v = (I+T)^{-1} f$, then

$\|v\|^2 \leq \langle Tv, v \rangle + \langle v, v \rangle = \langle f, v \rangle \leq \|f\| \|v\|$.

$\Rightarrow \|(T+I)^{-1} f\| \leq \|f\|$

Now let $\{u_n\}_{n=1}^{\infty} \subset D(T)$ s.t. $u_n \to u$

and $Tu_n \to f$.

It suffices to show that $u \in D(A)$ and $Tu = f$.

Observe that

$$u_n + Tu_n \to u + f$$

$$\Rightarrow \quad u_n = (I+T)^{-1}(I+T)u_n \to (I+T)^{-1}(u+f)$$

$$\Rightarrow \quad u = (I+T)^{-1}(u+f)$$

$$\Rightarrow \quad Tu = f$$

(c) Goal: Prove that if $R(T-\lambda_0 I) = H$ for some $\lambda_0 < 0$, then $R(T-\lambda I) = H$ for every $\lambda < \lambda_0/2$.

Let $\lambda_0 < 0$ and suppose $R(T-\lambda_0 I) = H$.

Then by part (b), for every $f \in H$, $\exists! v \in D(T)$ such that

$$(T-\lambda_0 I)v = f.$$

Moreover, $\|(T-\lambda_0 I)^{-1}\| \leq 1/|\lambda_0|$.

For $\lambda < \lambda_0/2$, consider

$$Tu - \lambda u = f$$

$$\Leftrightarrow \quad Tu - \lambda_0 u = f + (\lambda - \lambda_0)u.$$

$$\Leftrightarrow \quad u = (T-\lambda_0 I)^{-1}(f + (\lambda-\lambda_0)u).$$

$\Rightarrow \quad S u := (T-\lambda_0 I)^{-1}(f + (\lambda-\lambda_0)u)$

is a contraction if $|\lambda-\lambda_0| < |\lambda_0|$ $\square$.

Def: (Yosida Approximation)

Let T be a maximal monotone operator.

For every $\lambda > 0$, set

$$J_\lambda = (I+\lambda T)^{-1}, \quad T_\lambda = \frac{1}{\lambda}(I - J_\lambda).$$

T_λ is the Yosida Approximation of T.

Prop: i.) $\|T_\lambda v\| \leq \|Tv\| \quad \forall v \in D(T), \lambda > 0$

ii.) $\lim_{\lambda \to 0} T_\lambda v = Tv$

iii.) $\langle T_\lambda v, v\rangle \geq 0 \quad \forall v \in H, \forall \lambda > 0$

iv.) $\|T_\lambda v\| \leq \frac{1}{\lambda}\|v\|, \quad \forall v \in H \; \forall \lambda > 0$

Pf. Algebra

A general theorem on evolution equations on Banach spaces.

Thm (Cauchy, Lipschitz, Picard)

Let $(X, \|\cdot\|)$ be a Banach space and let $F: X \to X$ be an L-Lipschitz map.

Then given any $u_0 \in X$, there exists a unique solution $u \in C^1([0,\infty); X)$ satisfying

$$(*) \begin{cases} \frac{d}{dt} u = F u(t) & \text{on } [0,\infty) \\ u(0) = u_0 \end{cases}$$

u_0 is called the initial data.

Pf: Existence:

It suffices to find $u \in C([0,\infty); X)$ satisfying

$$u(t) = u_0 + \int_0^t F(u(s))\, ds.$$

Consider the Banach space $(Y, \|\cdot\|_Y)$ where

$$Y = \left\{ u \in C([0,\infty); X) \;\middle|\; \sup_{t \geq 0} e^{-2Lt} \|u(t)\|_X < \infty \right\}$$

and $$\|u\|_Y := \sup_{t \geq 0} e^{-2Lt} \|u(t)\|_X .$$

Define the function $\Phi : Y \to Y$ by

$$\Phi(u(t)) = u_0 + \int_0^t F(u(s))\,ds.$$

Then

$$\|\Phi u - \Phi v\|_Y \leq \frac{L}{2L} \|u - v\|_Y = \frac{1}{2} \|u - v\|_Y .$$

$\Rightarrow$ Φ has a unique fixed point given u_0 which gives a solution to $(*)$.

Uniqueness

Let u, v be solutions to $(*)$.

and let $\varphi(t) := \|u(t) - v(t)\|_X$.

Then $$\varphi(t) \leq L \int_0^t \varphi(s)\,ds \qquad \forall t \geq 0.$$

$$\leq L \int_0^{t_0} \varphi(s)\,ds + \varepsilon .$$

$$\Rightarrow \quad \frac{d}{dt}\left(L\int_0^t \varphi(s)\,ds + \varepsilon\right)$$

$$= L\varphi(t) \leq L^2\int_0^t \varphi(s)\,ds + \varepsilon L.$$

and $$\frac{d}{dt}\log\left(L\int_0^t \varphi(s)\,ds + \varepsilon\right)$$

$$= \left(L\int_0^t \varphi(s)\,ds + \varepsilon\right) \cdot L\varphi(t)$$

$$\leq L.$$

$$\Rightarrow \quad \log\left(L\int_0^t \varphi(s)\,ds + \varepsilon\right)$$

$$\leq Lt + \log(\varepsilon).$$

$$\Rightarrow \quad \varphi(t) \leq \varepsilon e^{Lt} \quad \Rightarrow \varphi(t) \equiv 0. \ \square$$

Thm: (Hille–Yosida)

Let T be a maximal monotone operator. Given any $u_0 \in D(T)$ $\exists!$ function

$$u \in C^1([0,\infty);H) \cap C([0,\infty);D(T)).$$

satisfying

$$(**)\begin{cases} \dfrac{du}{dt} + Tu = 0 & \text{on } [0,\infty) \\ u(0) = u_0 \end{cases}$$

Moreover,

$$\|u(t)\|_H \le \|u_0\|_H \quad , \quad \left\|\frac{du}{dt}\right\|_H = \|Tu(t)\|_H \le \|Tu_0\|_H.$$

for all $t \geq 0$.

Pf:

Step 1/6 : Uniqueness.

Let u, v be solutions to $(**)$.

Then

$$\left\langle \frac{d}{dt}(u-v), u-v \right\rangle = -\langle T(u-v), u-v\rangle \leq 0.$$

and

$$\left\langle u-v, \frac{d}{dt}(u-v) \right\rangle = -\langle T^*(u-v), u-v\rangle \leq 0.$$

$$\Rightarrow \quad \frac{d}{dt}\| u(t) - v(t)\|_H^2 \leq 0$$

$$\Rightarrow \quad u(t) = v(t).$$

For Steps 2–6, consider $\lambda > 0$ and let u_λ be a solution to

$$(***) \quad \begin{cases} \frac{d}{dt} u_\lambda + T_\lambda u_\lambda = 0 & t \in [0,\infty) \\ u_\lambda(0) = u_0 \in D(T). \end{cases}$$

Step 2/6

Lemma: Let $\omega \in C^1([0,\infty); H)$ be a function satisfying

$$\frac{d}{dt}\omega + T_\lambda \omega = 0 \quad \text{on } t \in [0,\infty)$$

then $t \mapsto \|\omega(t)\|_H$ and $t \mapsto \|\frac{d\omega}{dt}(t)\|_H$
$= \|T_\lambda \omega(t)\|_H$

are nonincreasing.

Proof of lemma:

$\frac{d}{dt}\|\omega(t)\|^2 \leq 0$ since $\langle T_\lambda \omega, \omega\rangle \geq 0$.

and the argument of step 1.

$$\frac{d}{dt}\left\|\frac{d}{dt}\omega(t)\right\|_H^2 = \frac{d}{dt}\left\langle \frac{d}{dt}\omega, \frac{d}{dt}\omega(t)\right\rangle$$

$$= -\left\langle T_\lambda\left(\frac{d}{dt}\omega\right), \frac{d}{dt}\omega(t)\right\rangle - \left\langle \frac{d}{dt}\omega(t), T_\lambda\left(\frac{d}{dt}\omega(t)\right)\right\rangle$$

$$\leq 0.$$

$\square$

The lemma implies

$$\| u_\lambda (t) \|_H \leq \| u_0 \|_H \qquad \forall t \geq 0 \quad \forall \lambda > 0$$

$$\left\| \frac{du_\lambda}{dt} (t) \right\|_H = \| A_\lambda u_\lambda (t) \|_H \leq \| A u_0 \|$$

$$\forall t \geq 0, \ \lambda > 0.$$