Thm: Let $T \in K(H)$, $\dim(H) = \infty$, then

(a) $0 \in \sigma(T)$.

(b) One of the following holds:

i.) $\sigma(T) = \{0\}$

ii) $\sigma(T) \setminus \{0\}$ is a finite set.

iii.) $\sigma(T) \setminus \{0\}$ is a sequence converging to 0.

Pf: a.) If $0 \notin \sigma(T)$ then T is invertible.

$\Rightarrow$ $T T^{-1} = I$ is compact

$\Rightarrow$ $B_H(0,1)$ is pre-compact

$\Rightarrow$ H is finite-dimensional ↯.

b.) Let $n \in \mathbb{Z}_+$ and consider

$$\sigma_n := \sigma(T) \cap \{ \lambda \in \mathbb{C} \mid |\lambda| \geq \tfrac{1}{n} \}.$$

Goal: Show that σ_n is either finite or empty.

If σ_n was infinite, then σ_n would have a limit point in

$$\{ \lambda \in \mathbb{C} \mid \tfrac{1}{n} \leq |\lambda| \leq \|T\| \},$$

contradicting the previous lemma.

Self-Adjoint Compact Operators

(Also defined in Part 3).

Def: $T \in \mathcal{L}(H,H)$ is self-adjoint if $T = T^*$

Observation: $\langle Tf, f \rangle \in \mathbb{R}$ for all $f \in H$.

Def: Let $T \in \mathcal{L}(H,H)$. The numerical range of T, denoted $\omega(T)$, is defined by

$$\omega(T) := \{ \langle Tf, f \rangle \mid f \in H, \|f\|_H = 1 \}.$$
$$\subset \mathbb{C}.$$

Prop: Let $T \in \mathcal{L}(H,H)$. Then

$$\sigma(T) \subset \overline{\omega(T)}$$

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pf: Assume $\lambda \in \mathbb{C}$, $\lambda \notin \overline{\omega(T)}$ and let $\alpha := \text{dist}(\lambda, \omega(T)) > 0$

$\Rightarrow$ for all $f \in H$, $\|f\| = 1$,

$$|\langle Tf, f\rangle - \lambda| \geq \alpha$$

$\Rightarrow$ $|\langle Tu - \lambda u, u\rangle| \geq \alpha \|u\|^2 \quad \forall u \in H.$

Now, if $S := T - \lambda I$, then

$$|\langle Su, u\rangle| > \alpha \|u\|^2 \quad \text{for all } u \in H.$$

$\Rightarrow$ S is injective.

Also, $\langle Su, f\rangle = 0$ for all $u \in H$ iff $f = 0$.

$\Rightarrow$ $R(S)$ is dense in H.

Finally, if $\{g_n\}_{n\geq 1}^{\infty} \subset R(S)$ is Cauchy, then

$\exists \{u_n\} \subset H$ s.t. $Su_n = g_n$ and

$$|\langle S(u_n-u_m), u_n-u_m\rangle| \geq \alpha \|u_n-u_m\|^2$$

$$\Rightarrow \quad \|u_n-u_m\| \leq \frac{1}{\alpha}\|g_n-g_m\|.$$

$\Rightarrow$ $\{u_n\}$ converges $\Rightarrow$ $\exists u \in H$ such that

$$Su = \lim_{n\to\infty} g_n.$$

$\Rightarrow$ $R(S)$ is closed.

$\Rightarrow$ $R(S) = H$

Thus S is injective and surjective which implies $\lambda \in \rho(T)$.

$\square$.

<u>Note:</u> $\sigma(T) \subsetneq \omega(T)$ can occur:

<u>Ex:</u> $T: \mathbb{C}^2 \to \mathbb{C}^2$ $\quad T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

Then $\omega(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1/2\}$, $\sigma(T) = \{0\}$.

Prop: Let $T \in \mathcal{L}(H,H)$ be a self-adjoint operator.

Let $m := \inf \omega(T)$, $M := \sup \omega(T)$

Then $m, M \in \sigma(T)$

and $\|T\| = \max(|m|, |M|)$

Pf: $\langle Tf, f\rangle \le M$

$\Rightarrow \quad 0 \le \langle (MI - T)f, f\rangle = \langle f, (MI-T)f\rangle.$

If $\omega(f,g) := \langle (MI-T)f, g\rangle$, then $\omega : H \times H \to \mathbb{C}$ is a complex symmetric, bilinear function with $\omega(f,f) \ge 0$.

Therefore, we have a Cauchy-Schwarz identity for ω:

$$|\omega(f,g)| \le \omega(f,f)^{1/2}\, \omega(g,g)^{1/2} \quad \forall f,g \in H.$$

$\Rightarrow$

$$\|(M I - T) f\| = \sup_{\|g\|=1} |\omega(f,g)| \leq C\, \omega(f,f)^{1/2}$$

By supremum property of M, $\exists \{f_n\}_{n=1}^{\infty}$ such that $\|f_n\| = 1$,

$$\langle T f_n, f_n \rangle \longrightarrow M$$

$$\Longleftrightarrow \quad \langle T f_n - M f_n, f_n \rangle \to 0.$$

$$\Longrightarrow \quad \|(M I - T) f_n\| \to 0$$

Finally, suppose, by contradiction, that

$$(M I - T)^{-1} \in \mathcal{L}(H, H)$$

then $\quad f_n = (M I - T)^{-1} (M I - T) f_n \to 0.$

which contradicts $\|f_n\| = 1$ for all n.

Thus, $\quad M \in \sigma(T)$.

$\|T\| = \max(|m|, |M|)$ follows from Prop 3.