Corollaries

- If $T \in K(H)$ and $\lambda \in \mathbb{C} \setminus \{0\}$, then

  $N(T - \lambda I)$ is finite-dimensional

- If $N(T - \lambda I) = \{0\}$, then

  $T - \lambda I$ is surjective.

- $T - \lambda I$ invertible $\iff$ $T^* - \bar{\lambda} I$ invertible.

Spectra and Eigenvalues

Def: Let $T \in \mathcal{L}(H, H)$

The resolvent set, denoted by $\rho(T)$, is defined by

$$\rho(T) := \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is invertible}\}.$$

The spectrum, denoted by $\sigma(T)$, is defined by

$$\sigma(T) := \mathbb{C} \setminus \rho(T)$$

The eigenvalues of T, denoted $EV(T)$, is defined by

$$EV(T) := \{\lambda \in \mathbb{C} \mid N(T - \lambda I) \neq \{0\}\}.$$

Notes: $\cdot$ $EV(T) \subset \sigma(T)$.

- If T is compact, then

$$EV(T) \setminus \{0\} = \sigma(T) \setminus \{0\}.$$

- $EV(T) \subsetneq \sigma(T)$ can hold.

For example, consider right-shift operator on $\ell^2(\mathbb{N})$.

Thm: Let $T \in \mathcal{L}(H,H)$. Then $\sigma(T)$ is compact and

$$\sigma(T) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}.$$

Pf: Suppose $\lambda \in \mathbb{C}$, $|\lambda| > \|T\|$

Goal: Show that $T - \lambda I$ is invertible.

In particular, we show that $T - \lambda I$ is bijective.

Let $f \in H$. Consider the equation

$$(*) \quad Tu - \lambda u = f.$$

Claim: $\exists! u \in H$ that satisfies $(*)$.

Pf of claim: $Tu - \lambda u = f \Leftrightarrow u = \frac{1}{\lambda}(Tu - f)$.

Let $F(u) = \frac{1}{\lambda}(Tu - f)$.

Then $\|F(u) - F(v)\| < \|u - v\|$

Contraction mapping principle $\Rightarrow \exists! u$ s.t. $F(u) = u$.

$\Rightarrow T - \lambda I$ is bijective.

Next, we prove that $\sigma(T)^c = \rho(T)$ is open.

Let $\lambda_0 \in \rho(T)$ and let $\lambda \in \mathbb{C}$ satisfy

$$|\lambda - \lambda_0| < \|(T - \lambda_0 I)^{-1}\|^{-1}.$$

Again, we want to show that $T - \lambda I$ is bijective.

Let $f \in H$,

$$Tu - \lambda u = f$$

$$\Leftrightarrow Tu - \lambda_0 u = f + (\lambda - \lambda_0)u.$$

$$\Leftrightarrow u = (T - \lambda_0 I)^{-1}(f + (\lambda - \lambda_0)u).$$

Now, if $\quad Su := (T-\lambda_0 I)^{-1}(f-(\lambda-\lambda_0)u)$.

then

$$\|Su - Sv\| = \|(T-\lambda_0 I)^{-1}(\lambda-\lambda_0)(u-v)\|$$
$$\leq \|(T-\lambda_0 I)^{-1}\| \, |\lambda-\lambda_0| \, \|u-v\|$$
$$< \|u-v\|.$$

$\Rightarrow$ $T-\lambda I$ is invertible $\Rightarrow$ $\sigma(T)$ is closed.

In conclusion, $\sigma(T)$ is compact $\square$.

<u>Lemma</u>: Let $T \in K(H)$ and let $(\lambda_n)_{n=1}^{\infty}$ be a sequence of distinct complex numbers such that

$$\lambda_n \to \lambda$$

and $\quad \{\lambda_n\}_{n=1}^{\infty} \subset \sigma(T) \setminus \{0\}$.

Then $\quad \lambda = 0$

(I.e. all points of $\sigma(T)\setminus\{0\}$ are isolated points).

Pf: Since T is compact, $\lambda_n \in EV(T)$.

For each n, choose $e_n \neq 0$ such that

$$(T - \lambda e_n) = 0.$$

Let $E_N = \text{span}\,\{e_n\}_{n=1}^{N}$

Claim: $E_N \subsetneq E_{N+1}$

proof of claim: by induction, assume $\{e_1, \dots, e_N\}$ are linearly independent.

Suppose $e_{N+1} = \sum_{j=1}^{N} \alpha_j e_j$

$$\Rightarrow Te_{N+1} = \sum_{j=1}^{N} \alpha_j Te_j = \sum_{j=1}^{N} \alpha_j \lambda_j e_j$$

and $Te_{N+1} = \lambda_{N+1} e_{N+1} = \sum_{j=1}^{N} \alpha_j \lambda_{N+1} e_j$

$$\Rightarrow 0 = \sum_{j=1}^{N} \alpha_j (\lambda_{N+1} - \lambda_j) e_j \Rightarrow \alpha_j(\lambda_j - \lambda_{N+1}) = 0 \quad \forall j$$

$$\Rightarrow \alpha_j = 0 \quad \forall j$$

$\Rightarrow \{e_1, \dots, e_{N+1}\}$ is lin. ind. $\Rightarrow E_N \subsetneq E_{N+1}$

Now construct a sequence, $u_n \in E_n$, s.t.

$$\|u_n\| = 1$$

and $\quad \operatorname{dist}(u_n, E_{n-1}) \geq 1/2.$

Then if $m > n \geq 2$, then $E_n \subset E_{m-1} \subset E_m$

$$T u_n / \lambda_n \in E_n \quad \text{and} \quad (T_m - \lambda_m I) u_m \in E_{m-1}.$$

which implies

$$\left\| T \frac{u_n}{\lambda_n} - T \frac{u_m}{\lambda_m} \right\| = \left\| T \frac{u_n}{\lambda_n} - T \frac{u_m}{\lambda_m} + u_m - u_m \right\|$$

$$= \left\| T \frac{u_n}{\lambda_n} - \frac{T u_m - \lambda_m u_m}{\lambda_m} - u_m \right\|$$

$$\geq \operatorname{dist}(u_m, E_{m-1}).$$

If $\lambda \neq 0$, then $\left\{ \frac{u_n}{\lambda_n} \right\}_{n \geq 1}^{\infty}$ is a bounded sequence

$\Rightarrow \left\{ T \frac{u_n}{\lambda_n} \right\}_{n \geq 1}^{\infty}$ is precompact because T is compact.

$\Rightarrow \left\{ T \frac{u_n}{\lambda_n} \right\}_{n \geq 1}^{\infty}$ has a Cauchy subsequence

Which would contradict $\left\| T\frac{u_n}{\lambda_n} - T\frac{u_m}{\lambda_m} \right\| \geq 1/2$

$\Rightarrow \lambda = 0$

$\square$.

<u>Thm:</u> Let $T \in K(H)$, $\dim(H) = \infty$, then

ⓐ $0 \in \sigma(T)$.

ⓑ One of the following holds:

i.) $\sigma(T) = \{0\}$

ii) $\sigma(T) \setminus \{0\}$ is a finite set.

iii.) $\sigma(T) \setminus \{0\}$ is a sequence converging to 0.

<u>Pf:</u> a.) If $0 \notin \sigma(T)$ then T is invertible.

$\Rightarrow T T^{-1} = I$ is compact

$\Rightarrow B_H(0,1)$ is pre-compact

$\Rightarrow H$ is finite-dimensional $\lightning$.