Therefore, $\exists$ subsequence $\{\varphi_{n_k}\}_{k=1}^{\infty}$ s.t.

$\{\varphi_{n_k}\}_{k=1}^{\infty}$ is Cauchy.

$$\Rightarrow \sup_{y \in B(0,1)} |\varphi_{n_k}(Ty) - \varphi_{n_l}(Ty)| \to 0$$

$$\Rightarrow \sup_{y \in B(0,1)} |\langle f_{n_k}, Ty\rangle - \langle f_{n_l}, Ty\rangle| \to 0$$

$$\Rightarrow \sup_{y \in B(0,1)} |\langle T^* f_{n_k}, y\rangle - \langle T^* f_{n_l}, y\rangle| \to 0.$$

$$\Rightarrow \|T^* f_{n_k} - T^* f_{n_l}\|_{\mathcal{H}} \to 0$$

$$\Rightarrow \{T^* f_{n_k}\} \text{ is Cauchy} \qquad \square.$$

<u>Thm:</u> (Fredholm Alternative)

Let $T \in K(H)$. Then

a.) $N(I-T)$ is finite-dimensional

b.) $R(I-T)$ is closed, and more precisely

$$R(I-T) = N(I-T^*)^{\perp}$$

c.) $N(I-T) = \{0\} \Leftrightarrow R(I-T) = \mathcal{H}$

d.) $\dim(N(I-T)) = \dim(N(I-T^*))$.

Pf:

a.) Let $F = N(I-T)$

$\Rightarrow B_F(0,1) \subset T(B_H(0,1))$

$\Rightarrow B_F(0,1)$ is pre-compact

$\Rightarrow$ F is finite-dimensional (Pset 2).

b.) Let $\{f_n\}_{n=1}^{\infty} \subset R(I-T)$, $f_n \to f$,

$f_n = u_n - Tu_n$

We want to show that $f \in R(I-T)$.

WLOG we can assume $u_n \notin N(I-T)$

and since $N(I-T)$ is a closed, linear subspace, we can write

$$H = N(I-T) \oplus N(I-T)^{\perp}$$

Moreover, we can decompose u_n into

$$u_n = u_n' + u_n'' \quad \text{where}$$

$$u_n' \in N(I-T)$$
$$u_n'' \in N(I-T)^{\perp}$$

This implies

$$f_n = u_n - Tu_n = u_n'' - Tu_n''$$

Therefore, WLOG we take $u_n \in N(I-T)^\perp$

Claim: $\{u_n\}$ is a bounded sequence

Proof of claim: If not, then

$$\|u_{n_k}\| \xrightarrow{k\to\infty} \infty \text{ for some subsequence.}$$

Since T is compact, there is a further subsequence, $\{u_{n_m}\}$, such that

$$T\left(\frac{u_{n_m}}{\|u_{n_m}\|}\right) \text{ converges, and}$$

$$f_{n_m}/\|u_{n_m}\| \xrightarrow{m\to\infty} 0.$$

Therefore,

$$\lim_{m\to\infty} \frac{u_{n_m}}{\|u_{n_m}\|} - T\left(\frac{u_{n_m}}{\|u_{n_m}\|}\right) = 0.$$

$$\Rightarrow \lim_{m\to\infty} \frac{u_{n_m}}{\|u_{n_m}\|} \in N(I-T)^\perp \cap N(I-T)$$

$$\Rightarrow \|u_{n_m}\|/\|u_{n_m}\| \to 0 \quad \text{which is a contradiction.}$$

Therefore, we can conclude that

$\{u_n\}$ is bounded.

Since T is compact, $\{Tu_n\}$ has a convergent subsequence, $\{Tu_{n_k}\}_{k=1}^{\infty}$.

Recall that $u_{n_k} = Tu_{n_k} + f_{n_k}$.

Thus u_{n_k} is the sum of two convergent sequences, which implies $\exists u$ s.t.

$$u_{n_k} \to u$$

Continuity of T implies

$$u - Tu = f \Rightarrow f \in R(I-T)$$

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c) $\underline{N(I-T) = \{0\}} \Rightarrow R(I-T) = H$

Assume $N(I-T) = \{0\}$, Suppose by contradiction, $R(I-T) =: H_0 \neq H$

$\Rightarrow$ H_0 is a Hilbert space

and $T(H_0) \subset H_0$

Then $T|_{H_0} \in K(H_0)$. and

$$R\left((I-T)|_{H_0}\right) = (I-T)(H_0) \subsetneq H_0 .$$

(Note: If $(I-T)(H_0) = H_0$, then $(I-T)(H_0) = (I-T)(H)$)

Let $$H_1 := (I-T)(H_0)$$

and $$H_n := (I-T)(H_{n-1}) = (I-T)^n(H).$$

Now $$H_{n+1} \subsetneq H_n \subsetneq H \qquad \text{for all } n \in \mathbb{N}.$$

Let $\{u_n\} \subset H$ be a sequence s.t.

$$u_n \in H_n, \quad \|u_n\| = 1, \text{ and } \operatorname{dist}(u_n, H_{n+1}) \geq 1/2.$$

Let $n > m$. Then

$$Tu_n - Tu_m = -(u_n - Tu_n) + (u_m - Tu_m) + u_n - u_m.$$

Observe: $-(u_n - Tu_n) + (u_m - Tu_m) + u_n \in H_{m+1}$

$\Rightarrow \|-(u_n - Tu_n) + (u_m - Tu_m) + u_n - u_m\| \geq \operatorname{dist}(u_m, H_{m+1}) \geq 1/2$

$\Rightarrow \quad \|Tu_n - Tu_m\| \geq 1/2$

$\Rightarrow \quad T(B_H(0,1))$ is not precompact, which is a contradiction.

($\Leftarrow$) $\underline{R(I-T)=H}$ $\Rightarrow N(I-T)=\{0\}$.

Recall that $T \in K(H) \Rightarrow T^* \in K(H)$.

Also, $R(I-T) = N(I-T^*)^{\perp}$. By assumption

$N(I-T^*) = \{0\}$. By reverse direction,

$N(I-T^*)=\{0\}$ implies $R(I-T^*) = H$.

Then $N(I-T)^{\perp} = R(I-T^*)$ concludes

the argument.

d.) $\underline{\text{Show}}$ that $\dim(N(I-T)) = \dim(N(I-T^*))$.

Let $d = \dim(N(I-T))$ and $d^* = \dim(N(I-T^*))$

Let $P : H \to N(I-T)$ represent the orthogonal projection onto $N(I-T)$ and let $L : N(I-T) \to N(I-T^*)$ be an injective, linear function.

Then $T - L \circ P$ is a compact operator

Moreover,

$$N(I - T + L \circ P) = \{0\}.$$

which implies, by part (c), that

$$R(I-T+L\circ P) = H.$$

$$\Rightarrow \quad R(I-T+L\circ P) = R(I-T) \oplus N(I-T^*).$$

$$\Rightarrow \quad R(I-T) + R(L\circ P) = R(I-T) \oplus N(I-T^*).$$

$\Rightarrow$ $R(L\circ P) = N(I-T^*)$

$\Rightarrow$ $L\circ P$ is surjective

$\Rightarrow$ $d^* \leq d$

Similarly, $d \leq d^*$

$\square$.