Cor: Let $g \in L^q(\mathbb{R}^d)$, let $\mathcal{B} \subset L^p(\mathbb{R}^d)$ be bounded, and $\Omega \subset \mathbb{R}^d$ have finite measure, then

$$\mathcal{F} := \{ (g * f)|_{\Omega} : f \in \mathcal{B} \}.$$

is precompact in $L^p(\Omega)$.

Before proving the corollary, we need a lemma:

Lemma: Let $g \in L^q(\mathbb{R}^d)$ for $1 \le q < \infty$. Then

$$\lim_{\|h\| \to 0} \| \tau_h g - g \|_q = 0.$$

Pf: Let $\varepsilon > 0$, $\exists g_0 \in C_c(\mathbb{R}^d)$ such that $\|g_0 - g\|_q < \varepsilon/3$. g_0 is uniformly continuous so

$$\| \tau_h g_0 - g_0 \|_q \le 2m(\operatorname{supp}(g_0))^{1/q} \| \tau_h g_0 - g_0 \|_\infty \xrightarrow{\|h\| \to 0} 0$$

$\Rightarrow$ for $\|h\|$ small enough

$$\underbrace{\| \tau_h g - g \|_q \le \| \tau_h g_0 - \tau_h g \|_q + \| \tau_h g_0 - g_0 \|_q + \| g - g_0 \|_q}_{< \varepsilon}$$

$\square$.

Proof of Corollary

It suffices to show that $\mathcal{F}$ is bounded and $\|\tau_h f - f\|_{L^p(\mathbb{R}^d)} \xrightarrow{\|h\|\to 0} 0$ uniformly in $\mathcal{F}$

Boundedness

For $f \in \mathcal{B}$, since $\mathcal{B}$ is bounded

$$\|(g * f)|_\Omega\|_p \le \|g * f\|_p \le \|g\|_1 \|f\|_p \le C\|g\|_1.$$

$\Rightarrow$ $\mathcal{F}$ is bounded in $L^p(\mathbb{R}^d)$

Convergence

$$\|\tau_h(g*f)|_\Omega - (g*f)|_\Omega\|_p \le \|\tau_h(g*f) - g*f\|_p$$

$$= \|(\tau_h g)*f - g*f\|_p$$

$$= \|(\tau_h g - g)*f\|_p$$

$$\le \|f\|_p \|\tau_h g - g\|_1$$

$$\le C\|\tau_h g - g\|_1$$

$\|\tau_h g - g\|_1$ converges to 0 indepent of $f \in \mathcal{B}$ $\square$

Compact Operators

Let X, Y be Banach spaces.

Def: (Compact Operators)

$T \in \mathcal{L}(X,Y)$ is compact if $T(B_X(0,1))$ is precompact in Y (norm topology).

We let $K(X,Y) := \{T \in \mathcal{L}(X,Y) \mid T \text{ compact}\}$.

$K(X) := \{T \in \mathcal{L}(X,X) \mid T \text{ compact}\}$.

Example:

Let $g \in L^1(\mathbb{R}^d)$. By previous corollary, the operator, $T : L^p(\Omega) \to L^p(\Omega)$, defined by

$$TF := (g * F)|_{\Omega}$$

is a compact operator.

Thm: $K(X,Y)$ is a closed linear subspace of $\mathcal{L}(X,Y)$.

pf:

① $K(X,Y)$ is linear: obvious

② $K(X,Y)$ is closed:

Let $(T_n)_{n=1}^{\infty} \subset K(X,Y)$ s.t. $\|T_n - T\|_{\mathcal{L}(X,Y)} \to 0$

Let $\varepsilon > 0$, $\exists N$ such that

$$\|T_N - T\| < \varepsilon/2.$$

Since T_N is compact, $T_N(B_X(0,1))$ is totally bounded

$$\Rightarrow \quad T_N(B_X(0,1)) \subset \bigcup_{i=1}^{m} B_Y(y_i, \varepsilon/2).$$

$$\Rightarrow \quad T(B_X(0,1)) \subset \bigcup_{i=1}^{m} B_Y(y_i, \varepsilon)$$

$\Rightarrow$ $T(B_X(0,1))$ is totally bounded

$\square$.

Def: (Finite Rank Operator)

An operator $T \in \mathcal{L}(X,X)$ is said to be of **finite rank** if the range of T, $R(T)$, is finite-dimensional.

Cor: Let $(T_n)_{n=1}^{\infty}$ be a sequence of finite-rank operators and let $T \in \mathcal{L}(X,X)$ be such that $\|T_n - T\|_{\mathcal{L}(X,X)} \to 0$.

Then $T \in K(X)$.

Approximation of Compact Operators by Finite-Rank operators

Thm: Let H be a Hilbert space and $T \in K(H)$, then $\exists\, (T_n)_{n=1}^{\infty}$ finite-rank operators such that

$$\|T_n - T\|_{\mathcal{L}(H,H)} \to 0.$$

Pf: (Idea: Use orthogonal projections).

Let $F = \overline{T(B_H(0,1))}$

F is compact and thus totally bounded, so given $n \in \mathbb{Z}_+$, there exists

$\{f_1, \ldots, f_m\} \subset H$ such that

$$F \subset \bigcup_{j=1}^{m} B_H(f_j, \tfrac{1}{n})$$

Let $V_n = \operatorname{span}(f_1, \ldots, f_m)$ and define

$T_n := P_{V_n} \circ T$ where $P_{V_n} =$ orthogonal projection onto V_n.

Then T_n is finite rank and

$$\|T_n x - Tx\| \le \|T_n x - f_{j_0}\| + \|Tx - f_{j_0}\|$$

$$\le \|T_n x - P_{V_n} f_{j_0}\| + \frac{1}{n}$$

$$= \|P_{V_n} Tx - P_{V_n} f_{j_0}\| + 1/n$$

$$\le \|Tx - f_{j_0}\| + 1/n \le 2/n$$

$\forall x \in B_H(0,1)$

$\Rightarrow \|T_n - T\| \le 2/n$ $\square$.

Thm: If $T \in K(H)$, then $T^* \in K(H)$.

Pf: Let $\{f_n\}_{n=1}^{\infty} \subset B_H(0,1)$.

Claim: $\{T^* f_n\}$ has a convergent subsequence. Let

$$F := \overline{T(B_H(0,1))}.$$

Then F is a compact metric space.

Let $\mathcal{F} := \{\varphi_n : F \to \mathbb{C} \mid \varphi_n(x) := \langle f_n, x \rangle\}$

Then $\mathcal{F} \subset C(F)$.

Observe: $|\tau_h \varphi_n(x) - \varphi_n(x_0)| \leq \|h\|$

Therefore, $\mathcal{F}$ is a bounded, equicontinuous subset of $C(F)$. Since F is compact, $\mathcal{F}$ satisfies the hypotheses of Arzela-Ascoli

Therefore, $\exists$ subsequence $\{\varphi_{n_k}\}_{k=1}^{\infty}$ s.t.

$\{\varphi_{n_k}\}_{k=1}^{\infty}$ is Cauchy.

$$\Rightarrow \sup_{y \in B(0,1)} |\varphi_{n_k}(Ty) - \varphi_{n_\ell}(Ty)| \to 0$$

$$\Rightarrow \sup_{y \in B(0,1)} |\langle f_{n_k}, Ty\rangle - \langle f_{n_\ell}, Ty\rangle| \to 0$$

$$\Rightarrow \sup_{y \in B(0,1)} |\langle T^* f_{n_k}, y\rangle - \langle T^* f_{n_\ell}, y\rangle| \to 0.$$

$$\Rightarrow \|T^* f_{n_k} - T^* f_{n_\ell}\|_{\mathcal{H}} \to 0$$

$$\Rightarrow \{T^* f_{n_k}\} \text{ is Cauchy} \qquad \square.$$

<u>Thm:</u> (Fredholm Alternative)

Let $T \in K(\mathcal{H})$. Then

a.) $N(I-T)$ is finite-dimensional

b.) $R(I-T)$ is closed, and more precisely

$$R(I-T) = N(I-T^*)^{\perp}$$

c.) $N(I-T) = \{0\} \iff R(I-T) = \mathcal{H}$

d.) $\dim(N(I-T)) = \dim(N(I-T^*))$.