Subspaces and Quotient Spaces

A subspace of X is a subset of X which is itself a linear space.

Given a subspace, M, of X, we define the equivalence relation

$$x \sim y \iff x - y \in M$$

Let $x+M$ denote the equivalence class containing x. Then

$$\alpha(x+M) + \beta(y+M) = (\alpha x + \beta y) + M.$$

Define

$$X/M := \{ x+M \mid x \in X \}.$$

If M is closed, then X/M inherits a norm from X:

$$\| x+M \| := \inf \{ \|x+y\| \mid y \in M \}.$$

Compact Sets in Banach Spaces.

In $\mathbb{R}^d$, compact $\Leftrightarrow$ closed and bounded.

For infinite-dimensional Banach Spaces, the equivalence doesn't necessarily hold.

Example:

$$\left(C([0,1]), \|\cdot\|_\infty \right)$$

$$\overline{B(0,1)} = \{ f \in C([0,1]) \mid \|f\|_\infty \le 1 \}.$$

Claim: $\overline{B(0,1)}$ is not compact.

Construct non Cauchy sequence.

Example: Equicontinuous families of functions.

Def: A subset, $\mathcal{F} \subset C(X)$, is <u>equicontinuous</u> at $x \in X$ if for all $\varepsilon > 0$, $\exists \delta > 0$ s.t. if $\rho(x,y) < \delta$ then $|f(x) - f(y)| < \varepsilon$ $\forall f \in \mathcal{F}$

Thm: (Arzela - Ascoli)

Let (X, ρ) be a compact metric space. If $\mathcal{F} \subset C(X)$ is equicontinuous and pointwise bounded, then

$\mathcal{F}$ is pre-compact (i.e. its closure is compact)

Proof by picture to show that $\mathcal{F}$ is totally bounded

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Separability

Examples.

① $$\ell^1(\mathbb{N}) := L^1(\mathbb{N}, \text{counting measure})$$

$$= \left\{ a_n : \mathbb{N} \to \mathbb{R} \;\middle|\; \sum_{n=1}^{\infty} |a_n| < \infty \right\}$$

$\ell^1(\mathbb{N})$ is separable.

② $$\ell^\infty(\mathbb{N}) = \left\{ a_n : \mathbb{N} \to \mathbb{R} \;\middle|\; \sup_n |a_n| < \infty \right\}$$

with norm $\|\{a_n\}\|_\infty := \sup_n |a_n|$

$\ell^\infty(\mathbb{N})$ is not separable.

Linear Operators

Def: A linear map, $T: X \to Y$, between two linear spaces, X and Y, satisfies

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

$$\forall \alpha, \beta \in \mathbb{R}, \quad x, y \in X$$

$[T(0) = 0]$.

Notes:

① A linear map, $T: X \to Y$, between two normed linear spaces, X and Y, is **bounded** if there exists $C \geq 0$ such that

$$\|Tx\|_Y \leq C \|x\|_X$$

for all $x \in X$

(Equivalently, $\exists C \geq 0$ s.t. $\displaystyle\sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx\|_Y \leq C$)