A Long Aside on Compact sets in Function Spaces

Arzela-Ascoli: (Compact subsets of $C(X)$)

Let (X,d) be a compact metric space.

Recall from earlier in the quarter:

Thm: If $\mathcal{F} \subset C(X)$ is equicontinuous and pointwise bounded in $C(X)$, then $\mathcal{F}$ is pre-compact.

L^p version of Arzela-Ascoli

Thm: Let $\mathcal{F}$ be a bounded set in $L^p(\mathbb{R}^d)$ with $1 \le p < \infty$. Assume that

$$\lim_{\|h\| \to 0} \|\tau_h f - f\|_p = 0 \quad \text{uniformly in } f \in \mathcal{F}$$

Then for any finite Lebesgue measure set, $\Omega \subset \mathbb{R}^d$,

$$\mathcal{F}|_{\Omega} = \{ f|_{\Omega} \mid f \in \mathcal{F} \} \text{ is precompact.}$$

Pf:

Goal: Find a precompact set of continuous functions that is dense in $\mathcal{F}|_{\Omega}$.

First, let $\{\Phi_N\}_{N=1}^{\infty}$ be a C_c^{∞} approximate identity. Then

$$\|f - \Phi_N * f\|_p = \left(\int \left| f(x) - \int f(x-y)\, \Phi_N(y) \right|^p \right)^{1/p}.$$

$$= \left(\int \left| f(x) \int \Phi_N(y) - \int f(x-y) \Phi_N(y) \right|^p \right)^{1/p}$$

By Jensen

$$\|f - \Phi_N * f\|_p \le C\left(\int\int |f(x) - f(x-y)|^p |\Phi_N(y)| dy\, dx\right)^{1/p}.$$

$$= C\left(\int |\Phi_N(y)| \int |f(x) - f(x-y)|^p dx\, dy\right)^{1/p}$$

$$= C\left(\int_{\|y\|>\delta} + \int_{\|y\|\le\delta} |\Phi_N(y)| \int |f(x) - f(x-y)|^p dx\, dy\right)^{1/p}$$

$$\le C\left(\|\Phi_N\|_1 \sup_{\|y\|<\delta} \|f - \tau_y f\|_p^p + 2\|f\|_p^p \int_{\|y\|\le\delta} |\Phi_N(y)| dy\right)^{1/p}$$

For δ small enough $\|f - \tau_y f\|_p^p < \varepsilon/2$ uniformly in $\mathcal{F}$ and for N large enough

$$\int_{\|y\|>\delta} |\Phi_N(y)| dy < \varepsilon/2.$$

$$\Rightarrow \quad \|f - \Phi_N * f\|_p \xrightarrow{N\to\infty} 0 \quad \text{unif for } f \in \mathcal{F}.$$

Next, we want to show that given $\varepsilon > 0$, $\Omega \subset \mathbb{R}$, $m(\Omega) < \infty$, there exists a compact set $E \subset \Omega$ such that

$$\|f\|_{L^p(\Omega \setminus E)} < \varepsilon \quad \text{for all } f \in \mathcal{F}.$$

This follows from the observation that

$$\|f\|_{L^p(\Omega\setminus E)} \le \|f - \Phi_N * f\|_{L^p(\mathbb{R}^d)} + \|\Phi_N * f\|_{L^p(\Omega\setminus E)}.$$

$$\le \frac{\varepsilon}{2} + \|\Phi_{N_0} * f\|_{L^\infty(\Omega\setminus E)} \cdot m(\Omega\setminus E)^{1/p}$$

$$\le \frac{\varepsilon}{2} + \|\Phi_{N_0}\|_{L^{p'}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)} \cdot m(\Omega\setminus E)^{1/p}$$

Therefore, it suffices to find a compact set $E \subset \Omega$ s.t. $m(\Omega\setminus E)^{1/p} \|\Phi_{N_0}\|_{L^{p'}} < \frac{\varepsilon}{2C}$.
for fixed N_0 large enough.
which would then imply

$$\|f\|_{L^p(\Omega\setminus E)} < \varepsilon.$$

Finally, we will show that $\mathcal{F}|_{\Omega}$ is totally bounded. Let $\varepsilon > 0$ and let $E \subset \Omega$ be a compact set satisfying

$$m(\Omega \setminus E) < \min\left(\varepsilon, \frac{\varepsilon^p}{C^p 2^p \|\Phi_{N_0}\|_{L^{p'}}^p}\right).$$

and

$$\mathcal{F}_0 := \left\{ (\Phi_{N_0} * f)|_E \;\middle|\; f \in \mathcal{F} \right\}.$$

Recall that N_0 is defined so that

$$\| \Phi_{N_0} * f - f \|_p < \varepsilon/2.$$

Observe:
$$\| \Phi_{N_0} * f - \tau_h(\Phi_{N_0} * f) \|_{L^\infty(E)}$$

$$= \| \Phi_{N_0} * (f - \tau_h f) \|_{L^\infty(E)}$$

$$\leq \| \Phi_{N_0} \|_{L^{p'}} \| f - \tau_h f \|_{L^p}$$

which implies that

$$\mathcal{F}_0 \subset C(E).$$

and $$\|\Phi_{N_0} * f\|_{L^\infty} \leq \|\Phi_{N_0}\|_{L^{p'}} \|f\|_{L^p}.$$

$\Rightarrow$ $\mathcal{F}_0$ is a bounded, equicontinuous family of continuous functions on a compact set, E.

$\Rightarrow$ $\mathcal{F}_0$ is totally bounded by Arzela-Ascoli

$\Rightarrow$ $\exists \{g_j\}_{j=1}^{M}$ such that for all $f \in \mathcal{F}$, $\exists j$ such that

$$\|\Phi_{N_0} * f - g_j\|_{L^\infty(E)} < \varepsilon.$$

$$\Rightarrow \quad \|\Phi_{N_0} * f - g_j\|_{L^p(E)} < m(E)^{1/p}\varepsilon < m(\Omega)^{1/p}\varepsilon.$$

$$\Rightarrow \|f - g_j\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega \setminus E)} + \|\Phi_{N_0} * f - f\|_{L^p(E)} + \|\Phi_{N_0} * f - g_j\|_{L^p(E)}.$$

$$\leq \varepsilon + \varepsilon + m(\Omega)^{1/p}\varepsilon \qquad \square$$

Cor: Let $g \in L^q(\mathbb{R}^d)$, let $\mathcal{B} \subset L^p(\mathbb{R}^d)$ be bounded, and $\Omega \subset \mathbb{R}^d$ have finite measure, then

$$\mathcal{F} := \{ (g * f)|_{\Omega} : f \in \mathcal{B} \}.$$

is <u>precompact</u> in $L^p(\Omega)$.

Before proving the corollary, we need a lemma:

<u>Lemma</u>: Let $g \in L^q(\mathbb{R}^d)$ for $1 \leq q < \infty$. Then

$$\lim_{\|h\| \to 0} \| \tau_h g - g \|_q = 0.$$

Pf: Let $\varepsilon > 0$, $\exists g_0 \in C_c(\mathbb{R}^d)$ such that $\|g_0 - g\|_q < \varepsilon/3$. g_0 is uniformly continuous so

$$\| \tau_h g_0 - g_0 \|_q \leq 2m(\operatorname{supp}(g_0))^{1/q} \| \tau_h g_0 - g_0 \|_\infty \xrightarrow{\|h\| \to 0} 0$$

$\Rightarrow$ for $\|h\|$ small enough

$$\| \tau_h g - g \|_q \leq \| \tau_h g_0 - \tau_h g \|_q + \| \tau_h g_0 - g_0 \|_q + \| g - g_0 \|_q < \varepsilon \quad \square.$$