Thm: Suppose $Tf = K * f$ is bounded on $L^2(\mathbb{R}^n)$ and K is a Calderón-Zygmund operator. Then $\exists\, C > 1$ such that

$$[Tf]_1 \leq C \|f\|_1 \quad \forall f \in L^1(\mathbb{R}^n).$$

Pf: It suffices to assume $f \in C_c^\infty(\mathbb{R}^n)$ since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.

Let $\lambda > 0$.

Perform a Calderón-Zygmund decomposition of f: $f = g + b$.

Now set

$$f_1 = g + \sum_{Q \in \mathcal{B}} \chi_Q \cdot ⨍_Q f \, dm$$

$$f_2 = b - \sum_{Q \in \mathcal{B}} \chi_Q \cdot ⨍_Q f \, dm$$

Let $f_Q = \chi_Q (f - ⨍_Q f)$, then

$$f_2 = \sum_Q f_Q.$$

where $f_Q := \chi_Q (f - ⨍_Q f)$

Observe:

- $f = f_1 + f_2$

- $\|f_1\|_\infty \leq 2^n \lambda$

- $\|f_1\|_1 \leq 2\|f\|_1$, $\|f_2\|_1 \leq 2\|f\|_1$.

Now

$$m(\{x \in \mathbb{R}^n \mid |Tf(x)| > \lambda\})$$

$$\leq m(\{x \in \mathbb{R}^n \mid |Tf_1(x)| > \tfrac{\lambda}{2}\}) + m(\{x \in \mathbb{R}^n \mid |Tf_2(x)| > \tfrac{\lambda}{2}\})$$

$$\leq \boxed{\frac{c}{\lambda^2}\|f_1\|_2^2}_{\,1} + \boxed{m(\{x \in \mathbb{R}^n \mid |Tf_2(x)| > \lambda/2\})}_{\,2}$$

For 1

$$\frac{c}{\lambda^2}\|f_1\|_2^2 \leq \frac{c}{\lambda^2}\|f_1\|_\infty \|f_1\|_1 \leq \frac{c}{\lambda}\|f_1\|_1 \leq \frac{c}{\lambda}\|f\|_1$$

For ②

For each $Q \in \mathcal{B}$, let y_Q denote the center of Q and let

$Q^* :=$ cube with center y_Q and diameter $2\sqrt{n} \cdot \operatorname{diam}(Q)$.

Then

$$m(\{|Tf_2| > \tfrac{\lambda}{2}\})$$

$$\leq m\Big(\bigcup_{Q \in \mathcal{B}} Q^*\Big) + m\Big(\Big\{x \in \mathbb{R}^n \setminus \bigcup_{Q\in\mathcal{B}} Q^* \;\Big|\; |Tf_2(x)| > \tfrac{\lambda}{2}\Big\}\Big)$$

$$\leq c \sum_{Q \in \mathcal{B}} m(Q) + \frac{2}{\lambda} \int_{\mathbb{R}^n - \bigcup_{Q\in\mathcal{B}} Q^*} |Tf_2(x)|\, dx$$

$$\leq C \frac{\|f\|_1}{\lambda} + \frac{2}{\lambda} \sum_{Q \in \mathcal{B}} \int_{\mathbb{R} \setminus Q^*} |Tf_2(x)|\, dx.$$

Observe: $$Tf_Q(x) = \int_Q K(x-y)\, f_Q(y)\, dm(y)$$

$$= \int_Q K(x-y)\, f_Q(y)\, dm(y) - K(x-y_Q) \int f_Q(y)\, dm(y)$$

$$= \int_Q \big(K(x-y) - K(x-y_Q)\big)\, f_Q(y)\, dm(y).$$

$\Rightarrow$

$$\int_{\mathbb{R}^n \setminus Q^*} |Tf_Q(x)|\, dm(x) \le \int_{\mathbb{R}^n \setminus Q^*} \int_Q |K(x-y) - K(x-y_Q)|\, |f_Q(y)|\, dm(y)\, dm(x)$$

$$= \int_Q |f_Q(y)| \int_{\mathbb{R}^n \setminus Q^*} |K(x-y) - K(x-y_Q)|\, dm(x)\, dm(y).$$

Recall condition (ii) of C-Z kernel:

$$\int_{\|x\| > 2\|y\|} |K(x) - K(x-y)|\, dx \le B \qquad \forall y \ne 0$$

This implies

$$\int_{\mathbb{R}^n \setminus Q^*} |K(x-y) - K(x-y_Q)|\, dm(x) \le B$$

Thus,

$$\int_{\mathbb{R}^n \setminus Q^*} |Tf_Q(x)|\, dm(x) \le B \int_Q |f_Q(y)| \le 2B \int_Q |f|.$$

$\Rightarrow$

$$\frac{2}{\lambda} \sum_{Q \in \mathcal{B}} \int_{\mathbb{R}^n \setminus Q^*} |Tf_Q| \le \frac{C}{\lambda} \sum_{Q \in \mathcal{B}} |f| \le \frac{C}{\lambda} \|f\|_1$$

$\square$.

The L^p boundedness of T.

Thm: (Calderón-Zygmund)
Let T be a Calderón-Zygmund Operator. Assume T is strong type $(2,2)$. Then T is strong type (p,p) for $1<p<\infty$.

pf: T is weak-type $(1,1)$.
and by assumption T is strong type $(2,2)$

Interpolation $\Rightarrow$ T is strong type (p,p) for $p \in (1,2)$.

Let $f \in C_c^\infty(\mathbb{R}^n)$, $g \in C_c^\infty(\mathbb{R}^n)$, $p \in (2,\infty)$
$q \in (1,2)$ and $\frac{1}{p}+\frac{1}{q}=1$.

Then

$$\int (Tf)\bar{g}\, dm = \langle Tf, g\rangle = \langle f, T^* g\rangle \quad \text{where}$$

$$T^*g(x) = \lim_{\varepsilon\to 0} \int_{\|x-y\|>\varepsilon} K^*(x-y)\, g(y)\, dy$$

where $K^*(x) = \overline{K(-x)}$

<u>Claim:</u> K^* is a Calderón-Zygmund kernel.

Therefore,

$$|\langle f, T^*g\rangle| \le \|f\|_p \|T^*g\|_q \le C\|f\|_p \|g\|_q.$$

$\square$.