Claim: $Tf = K*f$ is strong type (p,p) for $1<p<\infty$ and weak type $(1,1)$

First Step: T is strong type $(2,2)$.

Via Fourier Analysis

Lemma 1: $\|\hat{K}\|_\infty \le B$

Lemma 2: $\|Tf\|_2 = \|K*f\|_2 = \|\widehat{K*f}\|_2 = \|\hat{K}\hat{f}\|_2 \le \|\hat{K}\|_\infty \|f\|_2.$

Second Step: T is weak type $(1,1)$

Lemma: (Calderón - Zygmund Decomposition)

Let $f \in L^1(\mathbb{R}^n)$, $\lambda > 0$ we can write

$$f = g + b$$

where

- $|g(x)| \le \lambda$ for m-a.e. x
- $b = \sum_{Q \in \mathcal{B}} f \chi_Q$ where $\mathcal{B}$ is a collection of disjoint cubes
- if $Q \in \mathcal{B}$

$$\lambda < \frac{1}{m(Q)} \int_Q |f| \le 2^n \lambda$$

- $m\left(\bigcup_{Q\in\mathcal{B}} Q\right) < \frac{\|f\|_1}{\lambda}$

Pf: For each $k\in\mathbb{Z}$, define

$$\mathcal{D}_k = \{\text{dyadic cubes of length } 2^{k}\}.$$

Claim: $\exists k_0\in\mathbb{Z}$ such that

$$\frac{1}{m(Q)}\int_Q |f|\,dm \le \lambda \quad \text{for all } Q\in\mathcal{D}_{k_0}$$

Assuming the claim holds, for each of the 2^n children of Q, Q', either

$$\frac{1}{m(Q')}\int_{Q'}|f| \le \lambda \quad \text{or} \quad \frac{1}{m(Q')}\int_{Q'}|f| > \lambda.$$

If $\frac{1}{m(Q')}\int_{Q'}|f| > \lambda$, then Q' is a "bad" cube and we let $Q'\in\mathcal{B}$.

Observe: $\frac{1}{m(Q')}\int_{Q'}|f| \le \frac{2^n}{m(Q)}\int_Q |f| \le 2^n\lambda$

If $\frac{1}{m(Q')}\int_{Q'}|f| \le \lambda$, then we continue the process.

After this process is done, we have a collection of "bad" cubes, $\mathcal{B}$, where

$$\frac{1}{m(Q')} \int_{Q'} |f| > \lambda \iff \frac{1}{\lambda} \int_{Q'} |f| > m(Q')$$

$$\Rightarrow m\Big(\bigcup_{Q' \in \mathcal{B}} Q'\Big) \leq \frac{1}{\lambda} \int_{\cup Q'} |f| \leq \frac{\|f\|_1}{\lambda}.$$

For $x_0 \in \mathbb{R}^n \setminus \bigcup_{Q \in \mathcal{B}} Q$,

$\exists$ sequence of dyadic cubes $Q_m \to x_0$ such that

$$\frac{1}{m(Q_m)} \int_{Q_m} |f| \leq \lambda \quad \text{for all } m$$

Lebesgue differentiation $\Rightarrow$ $|f(x)| \leq \lambda$ m-a.e.

Finally, we let

$$g = f - \sum_{Q \in \mathcal{B}} f \chi_{\overline{Q}}$$

$\square$.

<u>Thm:</u> Suppose $Tf = K * f$ is bounded on $L^2(\mathbb{R}^n)$ and K is a Calderón-Zygmund operator. Then $\exists\, C > 1$ such that

$$[Tf]_1 \leq C \|f\|_1 \quad \forall f \in L^1(\mathbb{R}^n).$$

<u>pf.</u> It suffices to assume $f \in C_c^\infty(\mathbb{R}^n)$ since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.

Let $\lambda > 0$.

Perform a Calderón-Zygmund decomposition of f: $f = g + b$.

Now set

$$f_1 = g + \sum_{Q \in \mathcal{B}} \chi_Q \cdot ⨍_Q f \, dm$$

$$f_2 = b - \sum_{Q \in \mathcal{B}} \chi_Q \cdot ⨍_Q f \, dm$$

Let $f_Q = \chi_Q (f - ⨍_Q f)$, then

$$f_2 = \sum_Q f_Q.$$

where $f_Q := \chi_Q (f - ⨍_Q f)$

Observe:

- $f = f_1 + f_2$
- $\|f_1\|_\infty \le 2\lambda$
- $\|f_1\|_1 \le 2\|f\|_1$, $\|f_2\|_1 \le 2\|f\|_1$.

Now

$$m(\{x \in \mathbb{R}^n \mid |Tf(x)| > \lambda\})$$

$$\le m(\{x \in \mathbb{R}^n \mid |Tf_1(x)| > \tfrac{\lambda}{2}\}) + m(\{x \in \mathbb{R}^n \mid |Tf_2(x)| > \tfrac{\lambda}{2}\})$$

$$\le \underbrace{\boxed{\frac{C}{\lambda^2}\|f_1\|_2^2}}_{1} + \underbrace{\boxed{m(\{x \in \mathbb{R}^n \mid |Tf_2(x)| > \lambda/2\})}}_{2}$$

For 1

$$\frac{C}{\lambda^2}\|f_1\|_2^2 \le \frac{C}{\lambda^2}\|f_1\|_\infty \|f_1\|_1 \le \frac{C}{\lambda}\|f_1\|_1 \le \frac{C}{\lambda}\|f\|_1$$