(Note: weak type (p,∞) = strong type (p,∞).)

<u>Observe</u>: T strong type $(p,q) \Rightarrow T$ weak type (p,q).

<u>The Marcinkiewicz Interpolation Thm</u>

<u>Thm</u>: Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be measure spaces.

Let
$$p_0, p_1, q_0, q_1 \in [1,\infty].$$

$$p_0 \le q_0, \quad p_1 \le q_1 \quad \text{and} \quad q_0 \ne q_1$$

Define
$$\frac{1}{p} := \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q} := \frac{1-t}{q_0} + \frac{t}{q_1}$$

for $t \in (0,1)$.

If T is a sublinear map from $L^{p_0} + L^{p_1}$ to the space of measurable functions on Y that is both

weak type (p_0, q_0)

and weak type (p_1, q_1)

Then T is strong type (p,q).

We will not prove this.

Applications

Boundedness of Hardy-Littlewood Maximal Function.

Recall: For $f \in L^1_{loc}(\mathbb{R}^d)$

$$Mf(x) := \sup \left\{ \frac{1}{m(B)} \int_B |f| \, dm \;\middle|\; \begin{array}{c} x \in B \\ B \text{ is a ball} \end{array} \right\}.$$

Thm:

$$[Mf]_1 \leq C \|f\|_1$$

pf: 5r covering lemma.

Claim: $\cdot$ Mf is sublinear

$\cdot$ $\|Mf\|_\infty \leq C \|f\|_\infty$

Corollary to Marcinkiewicz Interpolation

The Hardy-Littlewood Maximal Function is strong-type (p,p) for $1 < p < \infty$.

Boundedness of Integral Operators

Let $\Omega \subset \mathbb{R}^{n+1}$ open,

$K : \bar{\Omega} \times \bar{\Omega} \setminus \{x=y\} \to \mathbb{C}$ be continuous

and $f \in L^p(\mathcal{H}^n|_{\partial\Omega})$

Finally, define

$$T_\delta f(x) := \int_{\partial\Omega \setminus B_\delta(x)} K(x,y)\, f(y)\, d\mathcal{H}^n(y)$$

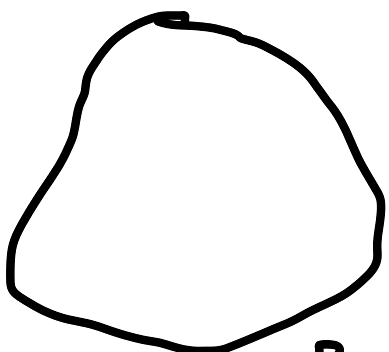
and $\quad Tf(x) := \lim_{\delta \to 0} T_\delta f(x).$

We would like to study the L^p boundedness of T.

Why? (From Folland's "Introduction to Partial Differential Equations")

Consider the following boundary value problem:

$$(*) \begin{cases} \Delta u(x) = 0 & , \ x \in \Omega \\ u(x) = f(x) & , \ x \in \partial\Omega \end{cases}$$



Ω.

Formally, demonstrating that there exists a function, $K(x,y)$ such that

$$\Delta_x K(\cdot,y) = 0$$
$$K(\cdot,x) = \delta_x \quad x \in \partial\Omega \quad \text{and}$$

$$u(x) = \int_{\partial\Omega} K(x,y) f(y) \, d\mathcal{H}^n$$

when $f \in C^2$ and $\partial\Omega$ is a C^2 surface.

However, u can be found using the method of "Layer Potentials"

Suppose $g \in L^p$ satisfies

$$u(x) = \int_{\partial\Omega} K(x,y) g(y) \, d\mathcal{H}^n \qquad x \in \Omega$$

$$f(x_0) = \lim_{z \to x_0} u(z) = \lim_{z \to x_0} \int_{\partial\Omega} K(z,y) g(y) d\mathcal{H}^n$$

for $x_0 \in \partial\Omega$.

Observe: (Nontrivial)

$$\lim_{z \to x_0} \int_{\partial\Omega} K(z,y) g(y) d\mathcal{H}^n = \frac{1}{2} g(x_0) + \int_{\partial\Omega} K(x_0,y) g(y) d\mathcal{H}^n$$

Thus, one can reduce the $(*)$ equation to the following set of equations

$$\begin{cases} \left(\frac{1}{2} I + T_1\right) g = f \\ \qquad u = T_2 g \end{cases}$$

where
$$T_1 g(x) = \int_{\partial\Omega} K(x,y)\, g(y)\, d\mathcal{H}^n(y) \qquad x \in \partial\Omega$$

$$T_2 g(x) = \int_{\partial\Omega} K(x,y)\, g(y)\, d\mathcal{H}^n(y) \qquad x \in \Omega$$

Therefore, it suffices to show that $\frac{1}{2} I + T_1$ is continuously invertible.

Thus, we need to understand the boundedness of T_1, then the invertibility of operators of the form $I + T$.

The case $K(x,y) = K(x-y)$

$$\Omega = \mathbb{R}^{n+1}_+ := \{ x_{n+1} > 0 \}$$

$$\partial\Omega = \mathbb{R}^n$$

Def: (Calderón-Zygmund Kernel).

Let $B > 0$, $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ satisfy

i.) $|K(x)| \leq B \|x\|^{-n}$

ii.) $\displaystyle\int_{\|x\| > 2\|y\|} |K(x) - K(x-y)| \, dx \leq B \quad \forall y \neq 0.$

iii.) $\displaystyle\int_{r < \|x\| < s} K(x) \, dx = 0 \quad \forall \; 0 < r < s < \infty$

Then K is called a <u>Calderón-Zygmund Kernel</u>.

<u>Example</u>: $K : \mathbb{R} \setminus \{0\} \to \mathbb{C}$, where $K(x) = \frac{1}{x}$

Lemma: For $f \in C_c^\infty(\mathbb{R}^n)$,

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{\|x-y\| > \varepsilon} K(x-y) f(y)\, dm(y)$$

exists.

pf: For $j \in \mathbb{Z}_+$

$$\left| \int_{2^{-j-1} \geq \|x-y\| > 2^{-j}} K(x-y) f(y)\, dm(y) \right|$$

$$= \left| \int_{2^{-j-1} \geq \|x-y\| > 2^{-j}} K(x-y) f(y)\, dy - f(x) \int_{2^{-j-1} \geq \|x-y\| > 2^{-j}} K(x-y)\, dy \right|$$

$$= \left| \int_{2^{-j-1} \geq \|x-y\| > 2^{-j}} K(x-y) \big(f(y) - f(x) \big)\, dy \right|.$$

$$\leq \|f'\|_\infty \int_{2^{-j-1} \geq \|x-y\| > 2^{-j}} |K(x-y)|\, \|x-y\|\, dy$$

$$\lesssim \|f'\|_\infty \cdot 2^{-jn} \cdot 2^{+j(n-1)} \lesssim 2^{-j} \|f'\|_\infty.$$